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Singular foliations and stability of fixed points in higher Lie theory

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Dissertation presented in partial fulfillment of the requirements for the degree of Doctor of Science (PhD): Mathematics

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Abstract

This thesis consists of three articles.

In the first article, we give examples of universal L_∞-algebroids, which algebraically desingularize singular foliations, of some foliations on a vector space V induced by linear Lie group actions. Universal L_∞-algebroids were introduced by Laurent-Gengoux-Lavau-Strobl. We then give a constructive way to directly compute some invariants of the singular foliation, defined by Laurent-Gengoux-Lavau-Strobl in a non-constructive way.

In the second article, we study the stability of fixed points of various geometric structures, extending the results of Crainic-Fernandes for fixed points, and the results of Dufour-Wade. We do this by showing that the stability problem for a fixed point is an instance of the following question about differential graded Lie algebras: given a differential graded Lie algebra $\mathfrak{g}$, a differential graded Lie subalgebra $\mathfrak{h}$, and a Maurer-Cartan element $Q \in \mathfrak{h}$ of degree 1, when do all Maurer-Cartan elements of $\mathfrak{g}$ near Q belong to $\mathfrak{h}$ up to gauge equivalence?

We then give a sufficient criterion for a positive answer to the question about differential graded Lie algebras when $\mathfrak{h}$ has degreewise finite codimension in $\mathfrak{g}$, and as application, we obtain stability criteria for fixed points of several geometric structures, such as Lie n-algebroids, Courant algebroids and Dirac structures admitting a complementary Dirac structure.

In the third article, we generalize the main result of the second article to $L_\infty[1]$-algebras and $L_\infty[1]$-subalgebras. This generalization then allows us to give a criterion for stability of fixed points of a Dirac structure, without needing a complentary Dirac structure as in the second article.

Beknopte samenvatting

Dit proefschrift bestaat uit drie artikelen.

In het eerste artikel geven we voorbeelden van universele L_∞-algebroïden, geïntroduceerd door Laurent-Gengoux-Lavau-Strobl, geassocieerd met singuliere foliaties die afkomstig zijn van een lineaire Lie groepswerking op een vectorruimte V. Universele L_∞-algebroïden zijn objecten die singuliere foliaties algebraïsch desingulariseren. Vervolgens geven we een constructieve methode om bepaalde invarianten van singuliere foliaties te berekenen, die door Laurent-Gengoux-Lavau-Strobl op niet-constructieve wijze gedefinieerd waren.

In het tweede artikel bestuderen we de stabiliteit van vaste punten van verschillende meetkundige structuren. Dit veralgemeniseert de resultaten van Crainic-Fernandes voor vaste punten, en de resultaten van Dufour-Wade. Om dit te bereiken, tonen we eerst aan dat het stabiliteitsprobleem voor vaste punten een voorbeeld is van het volgende vraagstuk over differentiaal gegradeerde Lie algebra's: zij $\mathfrak{g}$ een differentiaal gegradeerde Lie algebra, en $\mathfrak{h}$ een differentiaal gegradeerde Lie subalgebra. Zij Q een Maurer-Cartanelement van $\mathfrak{h}$ van graad 1. Wanneer zijn alle Maurer-Cartanelementen van $\mathfrak{g}$ nabij Q ijkequivalent met een element van $\mathfrak{h}$?

Vervolgens geven we een voldoende voorwaarde om de bovenstaande vraag over diferentiaal gegradeerde Lie algebra's positief te kunnen beantwoorden in het geval dat $\mathfrak{h}$ graadsgewijs eindige codimensie heeft in $\mathfrak{g}$. Als gevolg hiervan verkrijgen we stabiliteitscriteria voor vaste punten van verschillende meetkundige structuren, zoals Lie n-algebroïden, Courant algebroïden en Diracstructuren die een complementaire Diracstructuur toelaten.

In het derde artikel veralgemeniseren we het hoofdresultaat van het tweede artikel naar $L_\infty[1]$-algebra's en $L_\infty[1]$-subalgebra's. Deze veralgemenisering laat ons toe een voorwaarde te formuleren voor de stabiliteit van vaste punten van een Diracstructuur, zonder een complementaire Diracstructuur te vereisen.

Contents

Chapter 1

Introduction

This thesis contains three articles, written over the course of my PhD. These articles are all related to singular foliations, or aspects of their deformation theory. More precisely, we are concerned with the stability of fixed points, under deformations of various geometric structures that induce a singular foliation. In this introduction, we give a brief overview of the main results, provide the preliminaries and give an outline of the chapters.

1.1 Brief overview

1.1.1 Singular foliations

By a singular foliation on a manifold M, we mean a $C^\infty(M)$-submodule of the vector fields, closed under the Lie bracket of vector fields. Such a submodule induces a partition of M into connected, immersed submanifolds of different dimensions called *leaves*. As the submodule is generally not projective, it is not given by the sections of some vector bundle by the Serre-Swan theorem. Consequently, the number of local generators is not constant over M. For instance, let $M = \mathbb{R}^2$ with coordinates (x, y). Then for the foliation of all vector fields on $\mathbb{R}^2$ vanishing in the origin, any $p \neq 0$ admits an open neighborhood on which the foliation is generated by the vector fields ∂_x, ∂_y, while for any neighborhood of 0 there must be at least four generators: $x\partial_x, x\partial_y, y\partial_x, y\partial_y$.

One way to algebraically desingularize a singular foliation was provided in [LGLS20]. After picking a projective resolution of the module by sections of vector bundles, the authors define an algebraic structure on the projective resolution, which lifts the Lie bracket of the singular foliation. This lifted bracket however is no longer a Lie bracket, but satisfies the axioms of a Lie algebra in a weaker sense. The result of [LGLS20] is non-constructive, but the authors give plenty of examples. One of the difficulties lies in needing an projective resolution, which is already non-trivial for singular foliations induced by infinitesimal linear actions of Lie algebras on vector spaces. In particular, [LGLS20] contains the examples of the natural actions of $\mathfrak{sl}_2$ (traceless 2×2-matrices) on $\mathbb{R}^2$, $\mathfrak{so}_{p,q}$ (skew-symmetric matrices with respect to a symmetric bilinear form of signature (p, q)) on $\mathbb{R}^{p+q}$, $\mathfrak{gl}_n$ (all $n \times n$ matrices) on $\mathbb{R}^n$ and the action of a complex semisimple Lie algebra on itself by the adjoint action. Although these foliations are contained in the singular foliation induced by some $\mathfrak{gl}_n$ on $\mathbb{R}^n$, the restriction of a resolution to a submodule is not a resolution in general.

In Chapter 2, we present new examples of this construction for the following Lie subalgebras of $\mathfrak{gl}_n$ acting on $\mathbb{R}^n$:

- $\mathfrak{sl}_n$, the algebra of traceless endomorphisms, extending the result for $\mathfrak{sl}_2$,
- $\mathfrak{gl}_{n,k}$, the algebra of endomorphisms that preserve a fixed k-dimensional subspace,
- $\mathfrak{sp}_{2n}$, the algebra of endomorphisms of $\mathbb{R}^{2n}$ preserving the standard symplectic form
$$\begin{pmatrix} 0_n & -\mathrm{Id}_n \\ \mathrm{Id}_n & 0_n \end{pmatrix}$$
on $\mathbb{R}^{2n}$,

and their analogues for vector bundles.

We then give a constructive way to extract invariants out of a singular foliation, which were previously defined through the construction of [LGLS20], without needing to construct a projective resolution.

1.1.2 Stability

In Chapters 3 and 4, we discuss some aspects of the deformation theory of geometric structures inducing singular foliations. More precisely, we will consider singular foliations which are induced by geometric structures on vector bundles, and investigate the following question:

Given a geometric structure that induces a singular foliation on the manifold M, and a zero-dimensional leaf $p \in M$, when do all nearby geometric strucures of the same kind also have a zero-dimensional leaf near p?

In this case, the leaf is called stable. The stability question for Poisson manifolds and Lie algebroids was considered in [CF10] for general compact leaves and in [DW06] for higher order singularities, and the authors gave sufficient conditions for a positive answer.

Poisson manifolds are manifolds equipped with a bivector field $\pi \in \Gamma(\wedge^2 TM)$ satisfying an integrability condition. Zero-dimensional leaves of the underlying foliation then correspond to points $p \in M$ such that $\pi_p = 0$. On 2-dimensional manifolds, the integrability condition is vacuous.

On $M = \mathbb{R}^2$ with coordinates (x, y), consider for example the bivector fields $\pi_1 = (x^2 + y^2)\partial_x \wedge \partial_y$ and $\pi_2 = x\partial_x \wedge \partial_y$. Then the origin is a zero of both π_1 and π_2. It is stable as zero of π_2 by the intermediate value theorem, but not as zero of π_1: the bivector field

$$\pi^\epsilon = \pi_1 + \epsilon\partial_x \wedge \partial_y$$

is non-vanishing for $\epsilon > 0$.

Stability of a zero is necessary for the more natural question on *rigidity* of the Poisson structure: a Poisson structure π is called rigid in a neighborhood of a zero p, if any Poisson structure nearby is *isomorphic* to π on a (possibly smaller) neighborhood.

Rigidity is rare, and often difficult to show, requiring tools from analysis such as the Nash-Moser fast convergence method. Besides being a tool to disprove rigidity, there are instances where stability together with a normal form result implies rigidity. Indeed, the bivector field

$$\pi_{\mathfrak{so}_3} := x\partial_y \wedge \partial_z + y\partial_z \wedge \partial_x + z\partial_x \wedge \partial_y$$

is Poisson, and the origin is a stable zero. Moreover, for π' near $\pi_{\mathfrak{so}_3}$, it can be shown that if q is a point such that $\pi'_q = 0$, the first order approximation of π' around q is isomorphic to $\pi_{\mathfrak{so}_3}$. Finally, using the normal form result of [Con85], one can show that π' is isomorphic to its first order approximation, concluding rigidity of $\pi_{\mathfrak{so}_3}$.

The stability of a zero-dimensional leaf of a geometric structure does not only depend on the underlying singular foliation, but depends on the geometric structure. An instance of this is shown in [CF10]: the Poisson structure π_2 as above induces the foliation generated by $x\partial_x, x\partial_y$, with partition

$$\mathbb{R}^2 = \{x > 0\} \sqcup \{x < 0\} \sqcup \bigsqcup_{y \in \mathbb{R}} \{(0, y)\}.$$

Recall that the origin is a stable fixed point.

However, the infinitesimal action $\rho : \mathfrak{g} \to \mathfrak{X}(M)$ of the non-abelian 2-dimensional Lie algebra $\mathfrak{g} = \langle e_1, e_2 \mid [e_1, e_2] = e_2\rangle$ on $\mathbb{R}^2$ given by $\rho(e_1) = x\partial_x, \rho(e_2) = x\partial_y$ induces the same foliation, but the origin is not a stable leaf: for $\epsilon \neq 0$, the action $\rho_\epsilon : \mathfrak{g} \to \mathfrak{X}(M)$ given by

$$\rho_\epsilon(e_1) = x\partial_x + \epsilon\partial_y, \rho_\epsilon(e_2) = x\partial_y$$

does not have any zero-dimensional leaves, as $\rho_\epsilon(e_1)$ is a non-vanishing vector field.

Our goal is to extend the results of [CF10] and [DW06] to different geometric structures. To do this, we develop a systematic approach to address stability questions, making use of the algebraic framework underlying the deformation theory of the geometric structure, and obtain a sufficient condition for stability of zero-dimensional leaves. As application we can recover the known results of [CF10] for zero-dimensional leaves, and the results of [DW06], as well as various new results on stability for zero-dimensional leaves of:

i) higher Lie algebroids,

ii) Lie algebroid structures on vector bundles A, compatible with a fixed Lie algebroid structure on the dual vector bundle A^*,

iii) Poisson-Nijenhuis structures (which include holomorphic Poisson structures),

iv) Courant algebroids,

v) Dirac structures.

Using i), we also obtain a stability result for zero-dimensional leaves of singular foliations. Using a formal normal form result from [LGR21], we obtain a formal rigidity result for singular foliations, using a similar argument as for $\pi_{\mathfrak{so}_3}$ above. Moreover, the compatibility condition in ii) is the generalization of the condition that identifies Poisson structures amongst general Lie algebroid structures on the cotangent bundle.

1.2 Preliminaries

In this section we give some preliminaries. We define the objects that will appear throughout the majority of the thesis, such as singular foliations, Lie algebroids, Poisson structures, Lie n-algebroids, and we describe the tools we will use, such as graded geometry and L_∞-algebras, as well as what role they play in deformation theory.

1.2.1 Singular foliations

Let M be a smooth manifold. Intuitively, a singular foliation on M is a partition of M into immersed, connected submanifolds of varying dimension called *leaves*. Singular foliations appear naturally in differential geometry:

i) Any regular foliation can be viewed as a singular foliation for which the dimension of the leaves is constant on M.

ii) If G is a Lie group with an action on a smooth manifold M, then the orbits of the action give rise to a singular foliation on M.

iii) If $X \in \mathfrak{X}(M)$ is a vector field, then its orbits give rise to a singular foliation on M.

Examples ii) and iii) above are conceptually different. While example ii) directly describes the partition, example iii) only describes the directions tangent to the partition. To obtain the partition, the vector field X needs to be integrated. For regular foliations, the classical Frobenius theorem (see for instance [Lee13]) gives a condition for regular foliations when the two descriptions are equivalent:

Theorem (Frobenius)**.** *Let $D \subseteq TM$ be a vector subbundle. If $\Gamma(D)$ is closed under the Lie bracket, then there exists a partition $M = \bigsqcup_\alpha L_\alpha$ such that for every $p \in M$, $D_p = T_pL_\alpha$, where L_α is the unique leaf containing p.*

In other words, the Frobenius theorem states that regular foliations are in bijection with Lie subalgebras of the Lie algebra $\mathfrak{X}(M)$ of vector fields, that are the sections of some vector subbundle $D \subseteq TM$.

For singular foliations, the situation is more subtle, a historical overview can be found in [Lav18]. One difference is that the partition of M does not uniquely determine a Lie subalgebra: on $M = \mathbb{R}$, the orbits of the vector fields $X_k = x^k\partial_x$ are identical for each $k \geq 1$, while there is no diffeomorphism of M mapping $\mathcal{F}_k = \{fX_k \mid f \in C^\infty(M)\}$ to $\mathcal{F}_l = \{fX_l \mid f \in C^\infty(M)\}$ for $k \neq l$. As such, we define singular foliations taking the Lie subalgebra point of view, which is close to the definition that was previously used in [Her62, Ste74, Sus73].

Definition 1.2.1 ([LGLS20]). Let M be a smooth manifold. A *singular foliation* on M is a subsheaf[1] $\mathcal{F}$ of C^∞_M-modules of the sheaf $\mathfrak{X}_M$ of vector fields on M, which is

i) locally finitely generated,

ii) for every $U \subseteq M$ open, we have $[\mathcal{F}(U), \mathcal{F}(U)] \subseteq \mathcal{F}(U)$.

Proof of the equivalence between the definitions can be found in [Wan17]. It turns out that with this definition, a singular foliation induces a partition of M into immersed submanifolds, called *leaves*:

Theorem ([Her62]). *A singular foliation $\mathcal{F}$ induces a partition of M into leaves*

$$M = \bigsqcup_{\alpha} L_\alpha,$$

such that for L_α the unique leaf through p, $T_pL_\alpha = span_{\mathbb{R}}\{X_p \mid X \in \mathcal{F}_x\}$, where $\mathcal{F}_x$ is the germ of $\mathcal{F}$ at x.

This result builds a bridge between the algebraic Definition 1.2.1, and the intuitive geometric description given at the beginning of this section.

While Definition 1.2.1 is quite flexible, often the notion is too general. Therefore, we will typically consider singular foliations for which the sheaf $\mathcal{F}$ is the image[2] of a bracket-preserving map $\rho : \mathcal{E} \to \mathfrak{X}_M$, where $\mathcal{E}$ is the sheaf of sections of a vector bundle $E \to M$ equipped with some binary bracket on its sections. In this case, conditions i) and ii) are automatically satisfied. We discuss a class of these in the next subsection.

Unlike regular foliations, there is no local form for singular foliations around a point. However, locally around every point $p \in M$ the foliation is a product of the vector fields on the leaf through p with a singular foliation which is transverse to the leaves, and has p as a zero-dimensional leaf, as shown in [AZ13, Proposition 1.4]. The foliation transverse to the leaves is independent of the point p chosen on the leaf, and will be referred to as the *transverse foliation.*

1.2.2 Lie algebroids, Poisson manifolds and Courant algebroids

Let M be a smooth manifold. In this subsection, we introduce some geometric structures which naturally induce singular foliations.

[1] As we work in the smooth category, we will usually not distinguish between sheaves and global sections.

[2] As we work in the smooth category, the image of ρ at the level of sections is a sheaf.

Lie algebroids

We first introduce *Lie algebroids*, which simultaneously generalize the tangent bundle of a manifold, and Lie algebras, and were introduced in [Pra67].

Definition 1.2.2. A *Lie algebroid over M* is a triple $(A, \rho, [-,-]_A)$, where

i) $A \to M$ is a vector bundle,

ii) $\rho : A \to TM$ is a vector bundle map called the *anchor*,

iii) $[-,-]_A$ is a Lie bracket on the space of sections of A,

such that for all $f \in C^\infty(M)$, $x, y \in \Gamma(A)$,

$$[x, fy]_A = \rho(x)(f)y + f[x,y]_A. \tag{1.1}$$

Equation (1.1), together with the Jacobi identity for $[-,-]_A$ implies that $\rho : \Gamma(A) \to \mathfrak{X}(M)$ is a Lie algebra map. Consequently, the sheaf $\mathrm{im}(\rho)$ at the level of sections defines a singular foliation on M.

Example 1.2.3.

- When $M = \{*\}$ is a point, then a Lie algebroid over M is simply a Lie algebra.
- For any smooth manifold M, the tangent bundle $(TM, \mathrm{id}, [-,-])$ is a Lie algebroid M, where $[-,-]$ is the Lie bracket of vector fields.
- Given a regular foliation, with underlying subbundle $D \subseteq TM$, let $\iota : D \to TM$ be the inclusion map. Then $\left(D, \iota, [-,-]\big|_{\Gamma(D)}\right)$ is a Lie algebroid.
- Let $\mathfrak{g}$ be a Lie algebra with Lie bracket $[-,-]_{\mathfrak{g}}$, and let $\rho : \mathfrak{g} \to \mathfrak{X}(M)$ be a Lie algebra homomorphism, which is an infinitesimal action on M. Then $(\mathfrak{g} \times M, \tilde{\rho}, [-,-]_\rho)$ is a Lie algebroid called the *action Lie algebroid*. Here, for $(v,p) \in \mathfrak{g} \times M$,
$$\tilde{\rho}(v,p) = \rho(v)(p) \in T_pM,$$
and $[-,-]_\rho$ is the unique extension of $[-,-]_{\mathfrak{g}}$ satisfying (1.1).

In Chapters 3 and 4 we will encounter more examples and ways to construct new Lie algebroids out of existing ones.

To any Lie algebroid, there is a naturally attached cohomology, which generalizes the de Rham cohomology of a manifold, and the Chevalley-Eilenberg cohomology of a Lie algebra.

Definition/Lemma 1.2.4. Let $(A, \rho, [-,-])$ be a Lie algebroid. The exterior algebra $\Gamma(\wedge^\bullet A^*)$ carries a differential,

$$d_A : \Gamma(\wedge^\bullet A^*) \to \Gamma(\wedge^{\bullet+1} A^*).$$

For $X_0, \dots, X_k \in \Gamma(A), \alpha \in \Gamma(\wedge^k A^*)$ we have

$$\begin{aligned} d_A(\alpha)(X_0, \dots, X_k) = &\sum_{i=0}^{k} (-1)^i \rho(X_i)(\alpha(X_0, \dots, \widehat{X_i}, \dots, X_k) \\ &- \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_k). \end{aligned}$$

Then $d_A^2 = 0$.

It was shown in [Vai97] that the data of a square-zero degree 1 derivation on $\Gamma(\wedge^\bullet A^*)$ is equivalent to a Lie algebroid structure on A.

Poisson manifolds

For a smooth manifold M, any *Poisson structure* induces a Lie algebroid structure on T^*M. A Poisson structure on M is a Lie algebra structure on the algebra of smooth functions $C^\infty(M)$, compatible with the multiplication, and in this generality were introduced in [Lic77].

Definition 1.2.5. Let M be a smooth manifold. A *Poisson structure* on M is a skew-symmetric $\mathbb{R}$-bilinear map

$$\{-,-\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M),$$

such that for $f, g, h \in C^\infty(M)$,

a)
$$\{f, \{g, h\}\} = \{\{f, g\}, h\} + \{g, \{f, h\}\}$$

b)
$$\{f, gh\} = \{f, g\}h + g\{f, h\}.$$

Example 1.2.6. We start with the example of the *standard Poisson bracket on* $\mathbb{R}^{2n}$. Let $(q^1, \dots, q^n, p_1, \dots, p_n)$ be coordinates on $\mathbb{R}^{2n}$. For $f, g \in C^\infty(M)$,

$$\{f, g\} := \sum_{i=1}^{n} \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} - \frac{\partial g}{\partial p_i} \frac{\partial f}{\partial q^i} \tag{1.2}$$

defines a Poisson structure on $\mathbb{R}^{2n}$.

Poisson structures are named after Siméon Denis Poisson who originally used the standard Poisson bracket on $\mathbb{R}^{2n}$ to construct new conserved quantities out of existing ones in the Hamiltonian formulation of classical mechanics. Indeed, consider the physical system described by an energy function

$$H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

depending on the position $q = (q^1, \dots, q^n) \in \mathbb{R}^n$ and momentum $p = (p_1, \dots, p_n) \in \mathbb{R}^n$ of a particle moving in $\mathbb{R}^n$. Then Hamilton's equations of motion read

$$\begin{cases} \frac{d}{dt} q^i(t) = \frac{\partial H}{\partial p_i}(q(t), p(t)) \\ \frac{d}{dt} p_i(t) = -\frac{\partial H}{\partial q^i}(q(t), p(t)) \end{cases} .$$

Using the Poisson bracket (1.2), the equations of motion can be rewritten as

$$\frac{d}{dt}(q(t), p(t)) = X_H(q(t), p(t)),$$

where $X_H = \{H, -\} \in \mathfrak{X}(M)$ is the *Hamiltonian vector field* of H. Note that a function $f \in C^\infty(\mathbb{R}^{2n})$ is conserved along the integral curves of X_H if and only if

$$\{H, f\} = 0.$$

Consequently, using condition a) for a Poisson structure, it follows that if $f, g \in C^\infty(M)$ are conserved, then so is $\{f, g\}$.

The standard Poisson bracket on $\mathbb{R}^{2n}$ has the property that for every $x \in \mathbb{R}^{2n}$, $T_x\mathbb{R}^{2n}$ is spanned by Hamiltonian vector fields. In general this is not the case, but the set of Hamiltonian vector fields generate a singular foliation, by means of a Lie algebroid canonically associated to a Poisson structure.

Let M be a manifold, and $\{-, -\}$ a Poisson structure on M. As $\{-, -\}$ is a derivation of the pointwise product in each entry, there exists a unique bivector field $\pi \in \Gamma(\wedge^2 TM)$ such that for all $f, g \in C^\infty(M)$, we have the equality

$$\{f, g\} = \pi(df, dg). \tag{1.3}$$

Note that this does not make use of condition a) in Definition 1.2.5. To encode this condition in terms of π, we need an algebraic structure on the set of multivector fields, which was introduced by in [Sch40].

Definition/Lemma 1.2.7. Let M be a smooth manifold. Let $\mathfrak{X}^\bullet(M) := \Gamma(\wedge^\bullet TM)$. There are $\mathbb{R}$-bilinear operations

$$[-, -]_{SN} : \mathfrak{X}^{k+1}(M) \times \mathfrak{X}^{l+1}(M) \to \mathfrak{X}^{k+l+1}(M),$$

called the *Schouten-Nijenhuis* bracket of multivector fields, uniquely determined by the conditions that for $f \in C^\infty(M)$, $X, Y \in \mathfrak{X}(M)$, $\Lambda_1 \in \mathfrak{X}^{k+1}(M)$, $\Lambda_2 \in \mathfrak{X}^{l+1}(M)$, we have

$$[X, f]_{SN} = X(f),$$

$$[X, Y]_{SN} = [X, Y],$$

$$[\Lambda_1, X \wedge \Lambda_2]_{SN} = [\Lambda_1, X]_{SN} \wedge \Lambda_2 + (-1)^k X \wedge [\Lambda_1, \Lambda_2]_{SN}$$

$$[\Lambda_1, \Lambda_2]_{SN} = -(-1)^{kl}[\Lambda_2, \Lambda_1]_{SN}.$$

Here, the bracket in the second equality is the Lie bracket of vector fields. For $\Lambda_1 \in \mathfrak{X}^{k_1+1}(M), \Lambda_2 \in \mathfrak{X}^{k_2+1}(M), \Lambda_3 \in \mathfrak{X}^{k_3+1}(M)$, the Schouten-Nijenhuis bracket satisfies

$$[[\Lambda_1, \Lambda_2]_{SN}, \Lambda_3]_{SN} = [\Lambda_1, [\Lambda_2, \Lambda_3]_{SN}]_{SN} - (-1)^{k_1 k_2}[\Lambda_2, [\Lambda_1, \Lambda_3]_{SN}]_{SN}.$$

Using the Schouten-Nijenhuis bracket, the Jacobi identity for a Poisson structure $\{-,-\}$ can be encoded in the bivector field π as follows:

Lemma 1.2.8 ([Lic77])**.** *Let $\pi \in \mathfrak{X}^2(M)$, and let $\{-,-\}$ be defined as in* (1.3)*. Then $\{-,-\}$ satisfies the Jacobi identity if and only if $[\pi, \pi]_{SN} = 0$. In this case π is called a* Poisson bivector field*.*

A Poisosn bivector field $\pi \in \mathfrak{X}^2(M)$ now naturally induces a Lie algebroid structure on T^*M: according to the text after Definition/Lemma 1.2.4 it is sufficient to give to give a square zero derivation on the graded algebra $\Gamma(\wedge^\bullet TM) = \mathfrak{X}^\bullet(M)$.

Lemma 1.2.9. *Let π be a Poisson bivector field. Then*

$$[\pi, -]_{SN} : \mathfrak{X}^\bullet(M) \to \mathfrak{X}^{\bullet+1}(M)$$

is a degree 1 square-zero derivation on the algebra of multivector fields. Consequently, π equips T^*M with a Lie algebroid structure with anchor

$$\pi^\# : T^*M \to TM$$

given by the contraction, and Lie bracket

$$[-,-]_\pi : \Omega^1(M) \times \Omega^1(M) \to \Omega^1(M)$$

uniquely determined by

$$[df, dg]_\pi = d(\pi(df, dg)).$$

Consequently, any Poisson structure on a manifold M induces a singular foliation on M. The leaves naturally carry closed, non-degenerate 2-forms induced by π, making them into *symplectic* manifolds, but this is not relevant for this thesis.

Example 1.2.10.

- For $M = \mathbb{R}^{2n}$, the standard Poisson bracket (1.2) has associated bivector field
$$\pi = \sum_{i=1}^{n} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i}.$$
The anchor $\pi^{\#} : T^*M \to TM$ induces an isomorphism of Lie algebroids, hence the foliation only has one leaf.

- More generally, any symplectic manifold (M, ω) is also a Poisson manifold: the inverse of the isomorphism
$$\omega^{\flat} : TM \to T^*M$$
defines a bivector field which is Poisson because $d\omega = 0$.

- On the other extreme, for any manifold M, $\pi = 0$ is a Poisson bivector.

- Let $(\mathfrak{g}, [-,-]_{\mathfrak{g}})$ be a Lie algebra. Then $[-,-]_{\mathfrak{g}}$ induces a Poisson structure $\{-,-\}_{\mathfrak{g}}$ on $M = \mathfrak{g}^*$: it is uniquely determined by the condition that for linear functions $v, w \in \mathfrak{g} = (\mathfrak{g}^*)^* \subseteq C^{\infty}(\mathfrak{g}^*)$, we have
$$\{v, w\}_{\mathfrak{g}} = [v, w]_{\mathfrak{g}}.$$
The Lie algebroid structure on $T^*\mathfrak{g}^* \cong \mathfrak{g}^* \times \mathfrak{g}$ is the Lie algebroid associated to the coadjoint representation of $\mathfrak{g}$ on $\mathfrak{g}^*$. Consequently, the orbits are the coadjoint orbits of an integrating Lie group.

Courant algebroids

A geometric structure which is now known as the *standard Courant algebroid* was first introduced in [Cou90], in order to study symplectic and Poisson geometry in a constrained setting. In [LWX97], the general definition appeared, and it was shown that the Drinfel'd double of a Lie bialgebroid carries a Courant algebroid structure. This generalized the construction of the Drinfel'd double of a Lie bialgebra in [Dd86].

Definition 1.2.11. A *Courant algebroid* over M is a quadruple $(E, \langle -,- \rangle, \rho, [\![-,-]\!])$, where

i) $E \to M$ is a vector bundle,

ii) $\langle -, - \rangle : E \otimes E \to \mathbb{R}$ is a symmetric, non-degenerate pairing,

iii) $\rho : E \to TM$ is a vector bundle map covering the identity,

iv) $[\![-, -]\!] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ is an $\mathbb{R}$-bilinear map,

such that for all $x, y, z \in \Gamma(E), f \in C^\infty(M)$, we have

a) $[\![x, [\![y, z]\!]]\!] = [\![[\![x, y]\!], z]\!] + [\![y, [\![x, z]\!]]\!]$,

b) $\rho([\![x, y]\!]) = [\rho(x), \rho(y)]$,

c) $[\![x, fy]\!] = \rho(x)(f)y + f[\![x, y]\!]$,

d) $\rho(x)\langle y, z\rangle = \langle [\![x, y]\!], z\rangle + \langle y, [\![x, z]\!]\rangle$,

e) $[\![x, x]\!] = \frac{1}{2}\rho^*(d\langle x, x\rangle)$.

The set of conditions contains redundancies, and it was pointed out in [Uch02] that conditions b) and c) are implied by the others.

Observe that condition b) implies that $\rho(\Gamma(E)) \subseteq \mathfrak{X}(M)$ is a singular foliation. While there are similarities between Courant algebroids and Lie algebroids, there is also one important difference: the bracket $[\![-, -]\!]$ is not skew-symmetric, and the symmetric part is measured by $\langle -, - \rangle$. We give some examples.

Example 1.2.12.

- Let M be a smooth manifold. Then $TM \oplus T^*M$ carries a Courant algebroid structure, which is the so-called *standard Courant algebroid* introduced in [Cou90]. For $X_1, X_2 \in \mathfrak{X}(M)$, $\xi_1, \xi_2 \in \Omega^1(M)$, the pairing is given by the standard pairing

  $$\langle X_1 + \xi_1, X_2 + \xi_2\rangle = \xi_2(X_1) + \xi_1(X_2),$$

  the anchor is given by the projection to TM, and the bracket is given by

  $$[\![X_1 + \xi_1, X_2 + \xi_2]\!] = [X_1, X_2] + \mathcal{L}_{X_1}\xi_2 - \iota_{X_2} d\xi_1.$$

  This construction can be carried out for an arbitrary Lie algebroid $(A, \rho, [-, -]_A)$, to obtain a Courant algebroid structure on $A \oplus A^*$ as a special case of [LWX97].

- Let $(\mathfrak{g}, [-,-]_{\mathfrak{g}})$ be a Lie algebra, and let $\langle -,- \rangle$ be an invariant non-degenerate symmetric pairing on $\mathfrak{g}$ (for example the Killing form if $\mathfrak{g}$ is semisimple). Then $(\mathfrak{g}, \langle -,- \rangle, 0, [-,-]_{\mathfrak{g}})$ is a Courant algebroid over a point $\{*\}$. Courant algebroids over a point $\{*\}$ are also called *quadratic Lie algebras.*

- Let M be a smooth manifold, let $(\mathfrak{g}, \langle -,- \rangle, [-,-]_{\mathfrak{g}})$ be a quadratic Lie algebra, and let $\rho : \mathfrak{g} \to \mathfrak{X}(M)$ be an infinitesimal action with the property that
$$\langle \rho^*(\alpha), \rho^*(\beta) \rangle \equiv 0,$$
for $\alpha, \beta \in \Omega^1(M)$. Here $\langle -,- \rangle$ is viewed as a pairing on $\mathfrak{g}^*$. Then by [LBM08] $(\mathfrak{g} \times M, \langle -,- \rangle, \tilde{\rho}, [\![-,-]\!]_\rho)$ is a Courant algebroid, where as for the action Lie algebroid, $\tilde{\rho}$ is the constant extension of ρ, and for $x, y \in \Gamma(M \times g)$
$$[\![x,y]\!]_\rho = [x,y]_\rho + \rho^* \langle dx, y \rangle.$$
Here $[-,-]_\rho$ is the action Lie algebroid bracket, and $dx \in \Omega^1(M) \otimes \mathfrak{g}$ is the differential of x, seen as a function $x : M \to \mathfrak{g}$.

While Courant algebroids are not Lie algebroids, restricting the structure maps to certain subbundles may yield Lie algebroids. Subbundles of maximal rank, to which the pairing of the Courant algebroid restricts to zero are of particular interest. In the remainder of this section, we assume that E has even rank, and that the pairing $\langle -,- \rangle$ is of split signature.

Definition 1.2.13 ([Cou90])**.** Let $(E, \langle -,- \rangle, \rho, [\![-,-]\!])$ be a Courant algebroid of rank $2n$ with pairing of signature (n,n).

- A rank n subbundle $L \subseteq E$ is *Lagrangian* if $\langle -,- \rangle\big|_L = 0$.

- A Lagrangian subbundle $L \subseteq E$ is a *Dirac structure* if
$$[\![\Gamma(L), \Gamma(L)]\!] \subseteq \Gamma(L).$$

Dirac structures encode several geometric structures.

Example 1.2.14. Let M be a smooth manifold, and let $E = TM \oplus T^*M$ be the standard Courant algebroid.

- The graph of a bundle map $A : TM \to T^*M$ is Lagrangian if and only if A is skew-symmetric, i.e. $A = \omega^\flat$, for $\omega \in \Omega^2(M)$. It is moreover a Dirac structure if and only if ω is closed.

- The graph of a bundle map $A : T^*M \to TM$ is Lagrangian if and only if A is skew-symmetric, i.e. $A = \pi^\#$, for $\pi \in \mathfrak{X}^2(M)$. It is moreover a Dirac structure if and only if π is a Poisson bivector.
- Let $D \subseteq TM$ be a subbundle. Then it is easy to see that $D \oplus \mathrm{Ann}(D)$ is a Lagrangian subbundle, where $\mathrm{Ann}(D) := \{\alpha \in T^*M \mid \alpha|_D = 0\}$. It is a Dirac structure if and only if $\Gamma(D) \subseteq \mathfrak{X}(M)$, is closed under the Lie bracket, which by the Frobenius theorem is equivalent to D being tangent to a regular foliation.

Remark 1.2.15. A natural question is whether a Courant algebroid $(E, \langle -,-\rangle, \rho, [\![-,-]\!])$ whose pairing has split signature can be decomposed as the direct sum of two Dirac structures $E = L \oplus L'$. While this is can not always be done, given a Dirac structure $L \subseteq E$, one can always find a complementary *Lagrangian* subbundle L'.

1.2.3 L_∞-algebras

L_∞-algebras generalize Lie algebras by weakening the requirement of the Jacobi identity, and were introduced in [LS93]. L_∞-algebras will play two different roles in this thesis.

- In Chapters 2 and 3, we will use certain L_∞-algebras to algebraically desingularize singular foliations, as in [LGLS20], which we explain in Section 1.2.4.
- In Chapters 3 and 4, L_∞-algebras will be used to model deformations of geometric structures, which we explain in Section 1.2.6.

Graded linear algebra

In this section, we first give some background on graded linear algebra, establishing notation and the various conventions that will be used in the thesis.

Definition 1.2.16.

- A $\mathbb{Z}$*-graded vector space* is a vector space $V^\bullet$ with a decomposition $V^\bullet = \bigoplus_{i\in\mathbb{Z}} V^i$. An element $x \in V^i$ is called *homogeneous of degree* i, and its degree is denoted by $|x|$.
- A map $\phi : V^\bullet = \bigoplus_{i\in\mathbb{Z}} V^i \to W^\bullet = \bigoplus_{i\in\mathbb{Z}} W^i$ is of degree k if $\phi(V^i) \subseteq W^{i+k}$ for all $i \in \mathbb{Z}$.

- Let $V^\bullet = \bigoplus_{i\in\mathbb{Z}} V^i$ be a graded vector space. A subspace $W^\bullet \subseteq V^\bullet$ is a *graded subspace* if $W = \bigoplus_{i\in\mathbb{Z}} W \cap V^i$.

Remark 1.2.17. A graded vector space $V = \bigoplus V^i$ will be denoted by $V^\bullet$ if we want to emphasize the grading, or simply as V.

There are several ways to construct new graded vector spaces from existing ones.

Example 1.2.18.

- Let $k \in \mathbb{Z}$, and let $V^\bullet = \bigoplus V^i$ be a graded vector space. Then $V[k]^\bullet$ is again a graded vector space, with $V[k]^i = V^{i+k}$.

- Let $V^\bullet$ be a graded vector space, and let $W^\bullet \subseteq V^\bullet$ be a graded subspace. Then $(V/W)^\bullet$ is a graded vector space with $(V/W)^i = V^i/W^i = V^i/(W \cap V^i)$.

- Let $V^\bullet$ be a graded vector space. Then $(V^*)^\bullet := \mathrm{Hom}(V, \mathbb{R})$ is a graded vector space with $(V^*)^i = \mathrm{Hom}(V^{-i}, \mathbb{R})$.

- Let $V^\bullet$ and $W^\bullet$ be graded vector spaces. Then $(V \oplus W)^\bullet$ is a graded vector space, with
$$(V \oplus W)^i = V^i \oplus W^i.$$

- Let $V^\bullet$ be a graded vector space. Then $V^{\otimes k}$ is a graded vector space, with
$$(V^{\otimes k})^i = \bigoplus_{i_1+\cdots+i_k=i} V^{i_1} \otimes \cdots \otimes V^{i_k}$$

- Combining the previous two examples, consider for $V^\bullet$ a graded vector space, the *tensor algebra*:
$$T(V) := \bigoplus_{k\in\mathbb{Z}} V^{\otimes k}.$$
For $i, j \geq 0$, the natural concatenation map
$$V^{\otimes i} \otimes V^{\otimes j} \to V^{\otimes(i+j)}$$
extends to an algebra structure on $T(V)$.

- Let $V^\bullet$ be a graded vector space. Let $I_\pm(V) \subseteq T(V)$ be the two-sided ideal generated by elements of the form
$$v_1 \otimes v_2 \pm (-1)^{|v_1||v_2|} v_2 \otimes v_1$$

for $v_1, v_2 \in V$ homogeneous. Then there are two naturally associated quotient algebras:

$$\wedge V := T(V)/I_+,$$

$$S(V) := T(V)/I_-.$$

These quotient algebras are called the *exterior algebra of* V and the *symmetric algebra of* V respectively.

In the last example, there is a relation between the exterior algebra and the symmetric algebra, which is called the *décalage* isomorphism.

Lemma 1.2.19 (Décalage isomorphism)**.** *For every $k \geq 0$, the map*

$$S^k(V[1]) \to (\wedge^k V)[k],$$

$$v_1[1] \dots v_k[1] \mapsto (-1)^{\sum_{i=1}^{k} |v_i|(k-i)}(v_1 \wedge \cdots \wedge v_k)[k],$$

where the degree appearing in the exponent is the unshifted *degree of v_i, is an isomorphism of graded vector spaces. Moreover, the isomorphism is compatible with the multiplication, and induces an isomorphism of algebras*

$$S(V[1]) \cong \bigoplus_{k \geq 0} (\wedge^k V)[k].$$

We will use both the symmetric algebra and the exterior algebra interchangeably.

Differential graded Lie algebras

We start by defining a special case of L_∞-algebras, in which there are only two non-zero operations.

Definition 1.2.20. A *differential graded Lie algebra* is a triple $(\mathfrak{g}, \partial, [-,-])$, where

i) $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}^i$ is a graded vector space,

ii) $\partial : \mathfrak{g} \to \mathfrak{g}$ is a degree 1 map,

iii) $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a graded skew-symmetric map of degree 0,

such that

a) $\partial^2 = 0$,

b) For $x, y \in \mathfrak{g}$ homogeneous, we have

$$\partial([x,y]) = [\partial(x), y] + (-1)^{|x|}[x, \partial(y)].$$

c) For all $x, y, z \in \mathfrak{g}$ homogeneous, we have

$$[[x,y],z] = [x,[y,z]] - (-1)^{|x||y|}[y,[x,z]].$$

Here $|x|, |y|$ denote the degree of x and y respectively.

We give some examples.

Example 1.2.21.

- Any Lie algebra $(L, [-,-])$ can be viewed as a differential graded Lie algebra concentrated in degree 0, with $\partial = 0$.

- Any cochain complex (C, ∂) can be viewed as a differential graded Lie algebra with $[-,-] = 0$.

- Let $\left(A = \bigoplus_{i\in\mathbb{Z}} A^i, m\right)$ be a graded commutative $\mathbb{R}$-algebra. This means that (A, m) is an associative algebra, that the multiplication restricts to a map $m_{i,j} : A^i \times A^j \to A^{i+j}$, and that for $a \in A^i, b \in A^j$, we have $m(a,b) = (-1)^{ij}m(b,a)$ (such as $A = S(V)$, for some graded vector space V).

  For $k \in \mathbb{Z}$, an $\mathbb{R}$-linear map $\delta : A \to A$ is a *derivation of degree* k, if δ is a map of degree k, and for all $a \in A^i$, $b \in A^j$, we have

  $$\delta(m(a,b)) = m(\delta(a), b) + (-1)^{ik}m(a, \delta(b)).$$

  Let $\mathrm{Der}^k_{\mathbb{R}}(A)$ denote the set of derivations of degree k. Then

  $$\mathfrak{g}^\bullet = \bigoplus_{k\in\mathbb{Z}} \mathrm{Der}^k_{\mathbb{R}}(A)$$

  is a differential graded Lie algebra, with the graded commutator bracket $[-,-]$, which for $\delta_1 \in \mathrm{Der}^i_{\mathbb{R}}(A), \delta_2 \in \mathrm{Der}^j_{\mathbb{R}}(A)$ is defined by

  $$[\delta_1, \delta_2] = \delta_1 \circ \delta_2 - (-1)^{ij}\delta_2 \circ \delta_1.$$

  If $d : A \to A$ is a derivation of degree 1 such that $d^2 = 0$, then

  $$(\mathrm{Der}^\bullet_{\mathbb{R}}(A), [d,-], [-,-])$$

  is a differential graded Lie algebra.

- Let M be a smooth manifold. Then $(\mathfrak{g}^\bullet = \mathfrak{X}^{\bullet+1}(M), 0, [-,-]_{SN})$ is a differential graded Lie algebra with $\partial = 0$.

L_∞-algebras

We now give the general definition of an L_∞-algebra.

Definition 1.2.22. An L_∞-algebra is a pair $(\mathfrak{g}, \{\ell_k\}_{k\geq 1})$, where

- $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}^i$ is a graded vector space,
- $\ell_k : S^k(\mathfrak{g}[1]) \to \mathfrak{g}[1]$ are degree 1 linear maps called (multi)brackets,

such that for $n \geq 1$, we have

$$\sum_{i=1}^{n} \sum_{\sigma\in \mathrm{Sh}(i,\, n-i)} \epsilon(\sigma)\ell_{n-i+1}(\ell_i(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}) = 0. \tag{1.4}$$

Here $\epsilon(\sigma)$ is determined by the equation

$$\epsilon(\sigma)x_{\sigma(1)} \ldots x_{\sigma(n)} = x_1 \ldots x_n$$

in $S^n(\mathfrak{g}[1])$, and $\mathrm{Sh}(i, n-i)$ denotes the set of $(i, n-i)$ shuffles, which are permutations of $\{1, \ldots, n\}$ such that $\sigma(1) < \cdots < \sigma(i)$ and $\sigma(i+1) < \cdots < \sigma(n)$.

Equivalently, we say that $\mathfrak{g}[1]$ is an $L_\infty[1]$-*algebra* (see also [MZ12]).

Remark 1.2.23. By Lemma 1.2.19, we could equivalently define the L_∞-algebras using skew-symmetric brackets. If we write $\mu_k : \wedge^k \mathfrak{g}[k] \to \mathfrak{g}[1]$, then after applying degree shifts, we obtain maps $\mu_k : \wedge^k \mathfrak{g} \to \mathfrak{g}$ of degree $2-k$. Equation (1.4) is then equivalent to

$$\sum_{i=1}^{n} \sum_{\sigma\in \mathrm{Sh}(i,n-i)} \epsilon_{i,\sigma}\epsilon(\sigma)\mu_{n-i+1}(\mu_i(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}) = 0,$$

where $\epsilon_{i,\sigma} = (-1)^{i(n-i+1)} \cdot \mathrm{sgn}(\sigma)$.

Example 1.2.24.

- If $\ell_k \equiv 0$ for $k \geq 3$, then the conditions (1.4) are equivalent to the conditions for $(\mathfrak{g}, \ell_1, \ell_2)$ to be a differential graded Lie algebra, after applying the décalage isomorphism

$$S^2(\mathfrak{g}[1]) \cong (\wedge^2 \mathfrak{g})[2].$$

- If $\mathfrak{g} = \bigoplus_{i=-n+1}^{0}$, for degree reasons we necessarily have $\ell_k \equiv 0$ for $k > n+1$. In this case $(\mathfrak{g}, \{\ell_k\}_{1\leq k\leq n+1})$ is called a *Lie n-algebra*.

- Let $(E, \langle -,-\rangle, \rho, [\![-,-]\!])$ be a Courant algebroid. Then there is a naturally associated L_∞-algebra, with $\mathfrak{g}$ concentrated in degrees 0 and -1 described in [RW98]. Here, $\mathfrak{g}^0 = E$, and $\mathfrak{g}^{-1} = C^\infty(M)$. The multibrackets can be found in [RW98, Theorem 4.3].

In the next subsection we will encounter more examples, coming from singular foliations.

1.2.4 L_∞-algebroids

As discussed in Section 1.2.2, Lie algebroids induce singular foliations. However, the converse does not hold in general, see [AZ13, Lemma 1.3]. Moreover, even if the singular foliation $\mathcal{F}$ is induced by a Lie algebroid $(A, \rho, [-,-]_A)$, the Lie algebroid is not unique: for any Lie algebra $\mathfrak{g}$ with Lie bracket $[-,-]_\mathfrak{g}$ $(A \oplus \mathfrak{g} \times M, \rho \oplus 0, [-,-]_A \oplus [-,-]_\mathfrak{g})$ is a Lie algebroid with underlying foliation $\mathcal{F}$, where the bracket $[-,-]_\mathfrak{g}$ is extended fiberwise.

This question was addressed in [LGLS20], where the authors associate a higher Lie algebroid to a class of singular foliations. In many cases, the higher Lie algebroid is a Lie n-algebroid, which is in particular an L_∞-algebra concentrated in finitely many non-positive degrees:

Definition 1.2.25 ([Vor10])**.** Let M be a smooth manifold. Let $n \geq 1$. A Lie n-algebroid over M is a triple $(E, \rho, \{\ell_k\}_{1\leq k\leq n})$, where

i) $E = \bigoplus_{i=1}^n E_i[i-1] \to M$ is a graded vector bundle,

ii) $\rho : E_1 \to TM$ is a vector bundle map covering the identity,

iii) $\ell_k : \Gamma(S^k(E[1])) \to \Gamma(E[1])$ are $\mathbb{R}$-linear maps ($C^\infty(M)$-linear for $k \neq 2$) of degree 1,

such that for all $f \in C^\infty(M)$, $x, y \in \Gamma(E)$,

a)
$$\rho \circ \ell_1 = 0,$$
viewed as vector bundle maps $E_2 \to TM$.

b)
$$\ell_2(x, fy) = \rho(x)(f)y + f\ell_2(x, y),$$
where $\rho(x)$ is understood to be $\rho \circ \mathrm{pr}_{E_1}$.

c) For all $k \geq 1$,

$$\sum_{i=1}^{k} \sum_{\sigma \in \mathrm{Sh}(i,k-i)} \epsilon(\sigma) \ell_{k-i+1}(\ell_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(k)}) = 0.$$

The definition encompasses several known objects.

Example 1.2.26.

- A Lie 1-algebroid is equivalent to a Lie algebroid.
- A Lie n-algebroid for which $\ell_k \equiv 0$ for $k > 1$ is a cochain complex of vector bundles concentrated in degrees $-n+1, \ldots, 0$.
- A Lie n-algebroid for which $\ell_k \equiv 0$ for $k \geq 3$, is a differential graded Lie algebroid.
- Let $(C, \langle -, - \rangle, \rho, [\![-, -]\!])$ be a Courant algebroid over M. Then there is a Lie 2-algebroid associated to C: in this case, $E_1 = C$, $E_2 = T^*M$, and the multibrackets can be found in [JL19]. This Lie 2-algebroid is not canonically associated to C: it depends on a choice of connection on E, compatible with $\langle -, - \rangle$.

We now discuss a class of L_∞-algebroids, which can be associated to a singular foliation.

Universal L_∞-algebroids

In [LGLS20, Theorem 2.4], the authors show for a class of singular foliations $\mathcal{F}$ that over relatively compact subsets of M, $\mathcal{F}$ is induced by a Lie n-algebroid, for which the underlying cochain complex is a projective resolution of $\mathcal{F}$, algebraically desingularizing $\mathcal{F}$. Roughly, their procedure consists of two non-constructive steps:

1) Given the singular foliation $\mathcal{F}$, find a resolution

$$(\Gamma(E_\bullet), \partial) \xrightarrow{\rho} \mathcal{F} \to 0,$$

of $\mathcal{F}$ by sections of vector bundles $E_i \to M$,

2) Lift the Lie bracket on $\mathcal{F}$ to an L_∞-algebroid structure $(\rho, \{\ell_k\}_{k\geq 1})$. on $(\Gamma(E_\bullet), \partial)$ such that $\partial = \ell_1$.

We recall the precise definitions.

Definition 1.2.27 ([LGLS20]). Let $\mathcal{F}$ be a singular foliation. A *geometric resolution* of $\mathcal{F}$ is a triple $\left(E = \bigoplus_{i\geq 1} E_i[i], \partial, \rho\right)$, where

i) $E = \bigoplus_{i\geq 1} E_i[i]$ is a graded vector bundle,

ii) $\partial : E \to E$ is a vector bundle map of degree 1,

iii) $\rho : E_1 \to TM$ is a vector bundle map,

such that the sequence

$$\ldots \xrightarrow{\partial} \Gamma(E_n) \xrightarrow{\partial} \ldots \xrightarrow{\partial} \Gamma(E_1) \xrightarrow{\rho} \mathcal{F} \longrightarrow 0$$

is exact.

It is often convenient for the resolution to have finite length, i.e. $E_k = 0$ for $k \geq n$, for some $n \geq 0$. The following result gives a criterion for the existence of a finite length geometric resolution.

Proposition 1.2.28 ([LGLS20]). *Let $\mathcal{F}$ be a singular foliation on an n-dimensional manifold M such that for any $p \in M$, there exist coordinates $(x^1, \ldots, x^n)$ around p on a neighborhood U, such that $\mathcal{F}_p$ is generated by analytic vector fields in the coordinates $(x^1, \ldots, x^n)$. Then there exists a geometric resolution of $\mathcal{F}$ concentrated in degrees $-1, \ldots, -n$ over every relatively compact subset of M.*

The following result shows that the existence of a geometric resolution is sufficient for the existence of a L_∞-algebroid inducing the foliation.

Theorem 1.2.29 ([LGLS20]). *Let $\mathcal{F}$ be a singular foliation on M, and let (E, ∂, ρ) be a geometric resolution of $\mathcal{F}$. Then there exists an L_∞-algebroid structure $(\rho, \{\ell_k\}_{k\geq 1})$ on E, such that $\ell_1 = \partial$. Moreover, it is* universal*: for every other L_∞-algebroid $(E', \rho', \{\ell'_k\}_{k\geq 1})$ such that $\rho'(\Gamma(E'_1)) \subseteq \mathcal{F}$ there exists a morphism of L_∞-algebroids $E' \rightsquigarrow E$, which is unique up to homotopy*[3]*.*

In particular, for singular foliations which locally admit analytic generators, the universal L_∞-algebroid can be chosen to be a Lie n-algebroid. Several examples of universal Lie n-algebroids appear in Chapter 2.

[3]Morphisms of L_∞-algebroids and a homotopy of morphisms contain more data than cochain maps $E' \to E$ and a cochain homotopy between them. As the exact definitions are not relevant for the thesis, we omit them (see [LGLS20])

Let $\mathcal{F}$ be a singular foliation. Using a universal L_∞-algebroid, the authors associate to any point $p \in M$, the isotropy L_∞-algebra of $\mathcal{F}$ at p, which contains the isotropy Lie algebra of $\mathcal{F}$ at p.

Definition/Lemma 1.2.30 ([LGLS20])**.** Let $\mathcal{F}$ be a singular foliation, and $(E, \rho, \{\ell_k\}_{k\geq 1})$ a universal L_∞-algebroid. Let $p \in M$. Then the structure maps ℓ_k descend to cohomology $H^\bullet(E,p)$ of the complex

$$\dots \xrightarrow{\partial_p} (E_n)_p \xrightarrow{\partial_p} \dots \xrightarrow{\partial_p} (E_2)_p \xrightarrow{\partial_p} \ker(\rho_p) \longrightarrow 0 \ .$$

$H^\bullet(E,p)$ therefore inherits a L_∞-algebra structure, which, up to isomorphism does not depend on the choice of E. Moreover, $H^{-1}(E,p) := \ker(\rho_p)/\mathrm{im}(\partial_p : (E_2)_p \to (E_1)_p))$ is canonically isomorphic to the isotropy Lie algebra of $\mathcal{F}$ at p. The L_∞-algebra structure on

$$\bigoplus_{i\geq 1} H^{-i}(E,p)[i-1]$$

is called the *isotropy L_∞-algebra* of $\mathcal{F}$ at p.

In particular, the binary operation ℓ_2 defines a representation of $H^{-1}(E,p)$ on $H^{-i}(E,p)$, for $i \geq 1$. For $i = 1$, this is the adjoint representation.

1.2.5 Graded geometry

In this subsection we discuss graded geometry. Graded geometry provides a unified framework to describe Lie algebroids, Lie n-algebroids and Courant algebroids among other things. For an elaborate introduction, see for instance [CS11]. Graded geometry is a framework where, in addition to a commutative algebra of smooth functions, we allow functions that anti-commute. Such objects first appeared in physics, where anti-commuting variables were used to describe fermions.

Definition 1.2.31. A *non-negatively graded manifold* $(\mathcal{M}, C^\infty_\mathcal{M})$ *with base* M is a manifold M equipped with a sheaf of graded commutative algebras which locally trivialize as

$$C^\infty_\mathcal{M}|_U \cong \Gamma(S((E[1])^*))\big|_U \ ,$$

where E is a $\mathbb{Z}_{\leq 0}$-graded vector bundle (hence, $(E[1])^*$ is a $\mathbb{Z}_{\geq 1}$ graded vector bundle). The sheaf of algebras $C^\infty_\mathcal{M}$ will be referred to as the *sheaf of functions* on $\mathcal{M}$.

Remark 1.2.32. While $S((E[1])^*)$ is an infinite-dimensional vector bundle, it is degreewise finite-dimensional. A section $s : M \to S((E[1])^*)$ is considered smooth if s is a finite sum of homogeneous smooth sections.

Example 1.2.33. Let M be a smooth manifold.

- Any vector bundle A of rank n over M gives rise to a graded manifold over M, denoted by $A[1]$, with functions

$$C^\infty(E[1]) = \Gamma(S((A[1])^*)) \cong \Gamma\left(\bigoplus_{k=0}^{n} \wedge^k A^*[-k]\right).$$

  In particular, $T[1]M := TM[1]$ is a graded manifold, with

$$C^\infty(T[1]M) = \Omega^\bullet(M),$$

  the algebra of differential forms of M.

- More generally, any non-positively graded vector bundle E over M gives rise to a graded manifold denoted $E[1]$, with functions

$$C^\infty(E[1]) = \Gamma(S((E[1])^*)),$$

  It can be shown that any non-negatively graded manifold is of this form [BP13].

In practice, we are interested in the case where the algebra of functions carries a differential, making it into a differential graded commutative algebra.

Definition 1.2.34. Let $(\mathcal{M}, C^\infty_{\mathcal{M}})$ be graded manifold. The *vector fields on* $\mathcal{M}$ are given by

$$\mathfrak{X}(\mathcal{M}) := \mathrm{Der}_{\mathbb{R}}(C^\infty_{\mathcal{M}}),$$

the graded derivations of the algebra of functions. Note that the graded commutator turns $\mathfrak{X}(\mathcal{M})$ into a graded Lie algebra. A vector field Q of degree 1 is *cohomological* if $Q^2 = \frac{1}{2}[Q, Q] = 0$.

The choice of a cohomological vector field induces a differential on the graded algebra of functions of a graded manifold.

Definition 1.2.35. A *differential graded manifold* $(\mathcal{M}, C^\infty_{\mathcal{M}}, Q)$ is a graded manifold $(\mathcal{M}, C^\infty_{\mathcal{M}})$ equipped with a cohomological vector field $Q \in \mathfrak{X}^1(\mathcal{M})$.

We list some examples of differential graded manifolds.

Example 1.2.36.

- Let A be a vector bundle. On the graded manifold $A[1]$, any Lie algebroid structure gives rise to a cohomological vector field. Conversely, by [Vai97], the data of a cohomological vector field is equivalent to the data of a Lie algebroid structure. The vector field is precisely the differential of Definition/Lemma 1.2.4.
- Let $(E, \langle -, - \rangle)$ be a vector bundle over M with non-degenerate symmetric pairing $\langle -, - \rangle$. It was shown in [Roy02] that associated to this data, there is a graded manifold (non-canonically) isomorphic to $E[1] \oplus T^*M[2]$. The graded algebra of functions carries a degree -2 Poisson bracket, and a cohomological vector field which is moreover a derivation of this Poisson bracket is equivalent to a Courant algebroid structure.
- Let $n \geq 0$. Let E be a graded vector bundle concentrated in degrees $-n+1, \ldots, 0$. Then the data of a cohomological vector field on $E[1]$ is equivalent to a Lie n-algebroid structure on E [Vor10].

1.2.6 Deformations and stability

Deformation theory is the study of how properties change when considering a family of objects. The modern approach to deformation theory originates from the works of Kunihiko Kodaira and Donald Spencer on the deformations of complex manifolds [KS58], see for instance [Man22] for more details on the history.

As hinted in Section 1.2.3, the main tool to study deformation theory is L_∞-algebras. This idea follows the principle of Pierre Deligne, which was postulated in a letter written to John Millson [Del86]. Roughly, the principle states that every deformation problem is governed by a differential graded Lie algebra, with quasi-isomorphic differential graded Lie algebras describing equivalent deformation problems. More precisely, solutions to deformation problems correspond to certain elements in a differential graded Lie algebra or L_∞-algebra, which are so-called *Maurer-Cartan elements*. Moreover, under some conditions, there is a notion of equivalence called *gauge equivalence* on the Maurer-Cartan elements, which often corresponds to equivalences of solutions. While the notion of a deformation problem has been formalized, and the principle of Deligne has been made into a theorem in [Pri10, Lur10], we will mainly be interested in deformation problems in which solutions to the deformation problem carry an intrinsic topology, which will be clear from the context. As such, we will use this intrinsic topology to define deformations, and our constructions will be motivated by examples.

We first discuss the key ideas for differential graded Lie algebras, and then indicate the changes that need to be made for L_∞-algebras.

Deformation theory and differential graded Lie algebras

We discuss the notions necessary to do study deformations using differential graded Lie algebras. As mentioned earlier, we are interested in certain elements of differential graded Lie algebras.

Definition 1.2.37. Let $(\mathfrak{g}, \partial, [-,-])$ be a differential graded Lie algebra. Then $Q \in \mathfrak{g}^1$ is a *Maurer-Cartan* element if

$$\partial(Q) + \frac{1}{2}[Q, Q] = 0. \tag{1.5}$$

Maurer-Cartan elements can be used to modify the differential ∂, giving rise to a new differential graded Lie algebra structure on $\mathfrak{g}$:

Lemma 1.2.38. *Let $(\mathfrak{g}, \partial, [-,-])$ be a differential graded Lie algebra, and let $Q \in \mathfrak{g}^1$ be a Maurer-Cartan element. Then $(\mathfrak{g}, \partial + [Q,-], [-,-])$ is a differential graded Lie algebra.*

We compute the Maurer-Cartan elements of the differential graded Lie algebras discussed in Example 1.2.21.

Example 1.2.39.

- For a graded Lie algebra concentrated in degree 0, there are no Maurer-Cartan elements, as there are no elements of degree 1.
- For a cochain complex (C, ∂), Maurer-Cartan elements are 1-cocycles.
- Let $(A = \bigoplus_{i\in\mathbb{Z}} A^i, m)$ be a graded commutative $\mathbb{R}$-algebra. Then $\delta \in \mathrm{Der}^1_{\mathbb{R}}(A)$ is a Maurer-Cartan element if and only if $\delta^2 = 0$. Note that for $A = S(E^*[-1])$ for some non-positively graded vector bundle E over a manifold M, such δ encode Lie n-algebroid structures (see Example 1.2.36).
- For $\mathfrak{g}^\bullet = \mathfrak{X}^{\bullet+1}(M)$, $\pi \in \mathfrak{g}^1 = \mathfrak{X}^2(M)$ is Maurer-Cartan if and only if $[\pi, \pi]_{SN} = 0$, which is equivalent to π being a Poisson bivector field.

Often, solutions to deformation problems carry a natural notion of equivalence. To encode this equivalence, we need the notion of differentiable paths in $\mathfrak{g}^1$. In applications, $\mathfrak{g}^1$ will be the sections of some finite-dimensional vector bundle over M, so there is no problem. In more generality, the technical details for differential graded Lie algebras are listed in Assumptions 3.3.17. For simplicity, we do not elaborate on the details here.

Let $(\mathfrak{g}, \partial, [-,-])$ be a differential graded Lie algebra. The equivalences of Maurer-Cartan elements are induced by an "action" of $\mathfrak{g}^0$, which is an ordinary Lie algebra.

Definition 1.2.40. Let $(\mathfrak{g}, \partial, [-,-])$ be a differential graded Lie algebra. Let $X \in \mathfrak{g}^0$, and $Q \in \mathfrak{g}^1$. We define Q^X as the time 1 solution of

$$\frac{d}{dt}Q_t = \partial X - [X, Q_t]. \tag{1.6}$$

This is called the *gauge action of X on Q*.

Remark 1.2.41. One way to interpret this action is as follows, see for instance [Man22, Section 6.3]. Out of a differential graded Lie algebra $(\mathfrak{g}, \partial, [-,-])$ one can construct a new graded Lie algebra $(\mathfrak{g} \oplus \mathbb{R}\partial[-1], 0, [-,-]')$, such that $[\partial, x]' = \partial(x)$ for any $x \in \mathfrak{g}$ and $[\partial, \partial]' = 0$. Then the gauge action can be interpreted as the integration of the infinitesimal action of $\mathfrak{g}^0$ on the affine subspace $\{(x, \partial) \mid x \in \mathfrak{g}^1\} \subseteq \mathfrak{g}^1 \oplus \mathbb{R}\partial[-1]$.

We check what gauge equivalence corresponds to in the differential graded Lie algebras of 1.2.21.

Example 1.2.42.

- As for ordinary Lie algebras, there are no degree 1 elements, the action is trivial.
- In a cochain complex (C, ∂), for $c \in C^1$, $b \in C^0$, we have $c^b = c + \partial b$, so two cocycles are equivalent if they differ by a coboundary.
- Let $(A = \bigoplus_{i \in \mathbb{Z}} A^i, m)$ be a graded commutative $\mathbb{R}$-algebra. Then for $\delta \in \mathrm{Der}^1_{\mathbb{R}}(A)$, $X \in \mathrm{Der}^0_{\mathbb{R}}$, we have $\delta^X = \exp(X)\delta\exp(-X)$. Here $\exp(X)$ should be understood as integrating the infinitesimal automorphism of A associated to X. When $A = \Gamma(S(E^*[-1]))$ for some non-positively graded vector bundle E over a manifold M, this corresponds to integrating an infinitesimal automorphism of E.
- In $\mathfrak{g}^\bullet = \mathfrak{X}^{\bullet+1}(M)$, for $\pi \in \mathfrak{X}^2(M)$, $X \in \mathfrak{X}(M)$, we have $\pi^X = (\phi^X_{-1})_*\pi$, where ϕ^X_{-1} is the flow of X at time -1.

Above, we saw that some structures, including Lie (n-)algebroid structures and Poisson structures, which can be described as Maurer-Cartan elements in some differential graded Lie algebras, with isotopies corresponding to the gauge equivalence. For some structures, the algebraic structure encoding its deformations is not a differential graded Lie algebra, but an L_∞-algebra. The main example is that of Dirac structures, which we discuss in detail in Section 4.4.3.

Deformation theory and L_∞-algebras

In this section we generalize the definitions of Maurer-Cartan elements and gauge equivalence to L_∞-algebras. To avoid convergence issues, we assume that the L_∞-algebras have a finite amount of brackets. We choose to work with $L_\infty[1]$-algebras. Because of the degree shift, Maurer-Cartan elements live in degree 0.

Definition 1.2.43. Let $(\mathfrak{g}, \{\ell_k\}_{1\leq k\leq n})$ be an $L_\infty[1]$-algebra. An element $Q \in \mathfrak{g}^0$ is *Maurer-Cartan* if

$$\sum_{k=1}^{n} \frac{1}{k!}\ell_k(Q, \dots, Q) = 0.$$

The analogue of Lemma 1.2.38 holds, and the structure maps of an $L_\infty[1]$-algebra can be twisted to obtain a new $L_\infty[1]$-algebra. The proof can be found in for instance [Dol].

Lemma 1.2.44. *Let $(\mathfrak{g}, \{\ell_k\}_{1\leq k\leq n})$ be an $L_\infty[1]$-algebra, and let $Q \in \mathfrak{g}^0$. Define*

$$\ell_k^Q := \sum_{i=1}^{\infty} \frac{1}{i!}\ell_{i+k}(\underbrace{Q, \dots, Q}_{k \text{ times}}, -, \dots, -), \tag{1.7}$$

where we note that the sum is finite. If Q is Maurer-Cartan, then $(\mathfrak{g}, \{\ell_k^Q\}_{1\leq k\leq n})$ is an $L_\infty[1]$-algebra.

Using the formula for the twisted brackets, the equation for the gauge action can be written down concisely:

Definition 1.2.45. Let $(\mathfrak{g}, \{\ell_k\}_{1\leq k\leq n})$ be an $L_\infty[1]$-algebra. Let $X \in \mathfrak{g}^{-1}$, and $Q \in \mathfrak{g}^0$. We define Q^X as the time 1 solution of

$$\frac{d}{dt}Q_t = \ell_1^{Q_t}(X).$$

Remark 1.2.46. Observe that although L_∞-algebroids can be viewed as $L_\infty[1]$-algebras, they do not contain elements of degree 0, hence there are no non-trivial Maurer-Cartan elements.

Rigidity and stability

An important question in deformation theory is to determine conditions for an object Q_0 to be *rigid*, which means that every object Q near Q_0 of the same kind is equivalent to Q_0. The framework of differential graded Lie algebras allows to make this question concrete, using Maurer-Cartan elements to parametrize structures, and the gauge equivalence to capture equivalences. In ideal circumstances, rigidity should be implied by so-called *infinitesimal rigidity*.

Definition 1.2.47. Let $(\mathfrak{g}, \partial, [-,-])$ be a differential graded Lie algebra. Let $Q \in \mathfrak{g}^1$ be a Maurer-Cartan element. Then Q is *infinitesimally rigid* if

$$H^1(\mathfrak{g}, \partial + [Q,-]) = 0. \tag{1.8}$$

To explain this definition, we observe that rigidity of a Maurer-Cartan element $Q_0 \in \mathfrak{g}^1$ is equivalent to the map

$$\Phi : \mathfrak{g}^0 \to \mathfrak{g}^1, X \mapsto Q_0^X$$

being surjective onto an open neighborhood of Q_0 in the space of Maurer-Cartan structures. We claim that at the linear level, surjectivity is equivalent to (1.8).

Heuristically, the tangent space to the set of Maurer-Cartan elements should be the space of 1-cocycles in $(\mathfrak{g}, \partial + [Q,-])$, obtained by linearizing the Maurer-Cartan equation. At the level of tangent spaces, the map Φ then maps $X \in \mathfrak{g}^0$, to

$$\left.\frac{d}{dt}\right|_{t=0} Q_0^{tX} = \left.\frac{d}{dt}\right|_{t=0} Q_t = \partial X - [X, Q_0] = (\partial + [Q_0,-])(X),$$

where Q_t is the path associated to the gauge action of X on Q_0, proving the claim.

In the reasoning above, we made a lot of simplifications, and a number of things can go wrong:

- The set of Maurer-Cartan elements is generally not a smooth manifold,
- $\mathfrak{g}^1$ is generally infinite-dimensional.

This suggests that the implication

$$\text{infinitesimal rigidity} \implies \text{rigidity}$$

does not always hold. Nevertheless, we list a few cases in which this implication holds.

- Deformations of associative algebras [Ger64],
- Deformations of finite-dimensional Lie algebras [NR67],
- Under some additional assumptions, semilocal rigidity results for Poisson manifolds were obtained in [Măr14], extending local rigidity results in [MZ04].

Instead of rigidity, we can also consider *stability.* Given an object Q_0 with a property P, the property P is said to be *stable* if every Q near Q_0 has the property P up to equivalence. In terms of the differential graded Lie algebra $(\mathfrak{g}, \partial, [-,-])$ and Maurer-Cartan element Q_0, this means that we are interested in when the map

$$\Phi_P : \mathfrak{g}^0 \times \mathfrak{h}_P \to \mathfrak{g}^1, (X, Q) \mapsto (Q_0 + Q)^X$$

is surjective onto a open neighborhood of Q_0 in the space of Maurer-Cartan elements. Here $\mathfrak{h}_P := \{Q \in \mathfrak{g}^1 \mid Q + Q_0 \text{ is Maurer-Cartan and has property } P\}$. Note that for $\mathfrak{h}_P = 0$ we recover the rigidity problem. In this thesis, we are concerned with the case where $\mathfrak{h}_P$ is the set of Maurer-Cartan elements of a differential graded Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$.

Linearizing, it can be shown that infinitesimal surjectivity of Φ_P is equivalent to the map

$$H^1(\mathfrak{h}, \partial + [Q_0, -]) \to H^1(\mathfrak{g}, \partial + [Q_0, -]) \tag{1.9}$$

induced by the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ being surjective. In many cases, $\mathfrak{h}_P$ is the set of Maurer-Cartan elements of a differential graded Lie subalgebra $\mathfrak{h}$. The question can then be reformulated as follows:

Let $(\mathfrak{g}, \partial, [-,-])$ be a differential graded Lie algebra, and let $\mathfrak{h} \subseteq \mathfrak{g}$ be a differential graded Lie subalgebra. Let $Q_0 \in \mathfrak{h}^1$ be a Maurer-Cartan element. When is every Maurer-Cartan element of $\mathfrak{g}$ near Q_0 related to an element in $\mathfrak{h}^1$ by the gauge action?

Example 1.2.48. The motivating example for this question is the stability of leaves of Poisson/Lie algebroid structures as in [CF10]. A leaf $L \subseteq M$ of a Lie algebroid $(A, \rho, [-,-])$ over a manifold is stable if nearby Lie algebroid structures $(\rho', [-,-]')$ have a leaf diffeomorphic to L. In [CF10], the authors give a sufficient condition cohomological condition for the stability of a compact leaf. For zero-dimensional leaves, this condition is the vanishing of the cohomology group

$$H^1_{CE}(A_p, T_pM),$$

the first Lie algebra cohomology group of A_p with values in T_pM. In Section 3.3.2, we show that for a zero-dimensional leaf $\{p\}$, stability is in fact an instance of the question above. In this case, $\mathfrak{g} = \mathrm{Der}_{\mathbb{R}}(\Gamma(\wedge^\bullet A^*))$, and the property whose stability we investigate is that $\{p\}$ is a leaf. Hence, as differential graded Lie subalgebra, we set for $k \geq 0$

$$\mathfrak{h}^k := \{\delta \in \mathrm{Der}^k_{\mathbb{R}}(\Gamma(\wedge^\bullet A^*)) \mid \delta(C^\infty(M)) \subseteq I_p\Gamma(\wedge^k A^*)\}.$$

The subspace $\mathfrak{h} = \bigoplus_{k\geq -1} \mathfrak{h}^k$ is closed under the graded commutator bracket, so it defines a differential graded Lie subalgebra of $(\mathfrak{g}, 0, [-,-])$, and its Maurer-Cartan elements are Lie algebroid structures on A for which $\{p\}$ is a leaf. In these terms, the cohomological obstruction is isomorphic to

$$H^1(\mathfrak{g}/\mathfrak{h}, \overline{\partial + [Q,-]}), \tag{1.10}$$

the cohomology of the quotient complex. Observe that by the long exact sequence in cohomology associated to the short exact sequence of complexes,

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{h} \longrightarrow 0,$$

the vanishing of (1.10) implies the surjectivity of (1.9).

Remark 1.2.49. For a deformation problem that is encoded by an $L_\infty[1]$-algebra $(\mathfrak{g}, \{\ell_k\}_{1\leq k\leq n})$, the analogous observations can be made. The difference is now that the infinitesimal condition for rigidity of a Maurer-Cartan element $Q \in \mathfrak{g}^0$ is

$$H^0(\mathfrak{g}, \ell_1^Q) = 0,$$

where ℓ_1^Q is defined as in (1.7). Note that the use of the zeroth cohomology is solely due to the degree shift in $L_\infty[1]$-algebras.
For stability, we will consider properties that can be encoded in $L_\infty[1]$-subalgebras $\mathfrak{h} \subseteq \mathfrak{g}$, and the infinitesimal stability condition for $Q \in \mathfrak{h}^0$ is then the surjectivity of the map

$$H^1(\mathfrak{h}, \ell_1^Q) \to H^1(\mathfrak{g}, \ell_1^Q).$$

1.3 Outline of the chapters

We give an outline of the chapters.

1.3.1 Chapter 2

The second chapter is the article “On the universal L_∞-algebroid of linear foliations”, and is available on the arXiv with identifier arXiv:2207.03278, and will appear in the Journal of Lie Theory.

1.3.1.1 Main results

This chapter consists of two parts. In the first part, consisting of Sections 2.2 and 2.3, we compute several new examples of universal L_∞-algebroids as in [LGLS20]. More precisely, in Section 2.2, we construct a universal L_∞-algebroid for several singular foliations induced by linear Lie algebra actions on a vector space V, by explicitly constructing projective resolutions of the C^∞_V-module underlying the singular foliation, and then defining an L_∞-algebroid structure on the space of sections. The Lie algebras we consider are:

- $\mathfrak{gl}(V, W)$ for a subspace $W \subseteq V$, which consists of endomorphisms of V that preserve W, in which case we obtain a Lie n-algebroid with only a unary and binary bracket.
- $\mathfrak{sl}(V)$, which consists of traceless endomorphisms of V, in which case we again obtain a Lie n-algebroid with only a unary and binary bracket.
- $\mathfrak{sp}(V, \omega)$ for a non-degenerate $\omega \in \wedge^2 V^*$, which consists of endomorphisms of V that preserve ω. In this case only the partial L_∞-algebroid structure is found, and we show that there is a non-zero ternary bracket.

In Section 2.3, we consider foliations on vector bundles for which the zero section L is a leaf, and the transverse foliation on the fibers is given by one of the Lie algebras above. These foliations generalize the ones above: they are generated by vector fields whose flow is a vector bundle automorphism. Moreover, the flow of the generating vector fields preserve a subbundle, a fiberwise volume form and a fiberwise symplectic form respectively.

In the second part, which is Section 2.4, we give a constructive way to compute some of the invariants defined in [LGLS20]. More specifically, given a foliation $\mathcal{F}$ on a manifold M and a zero-dimensional leaf $p \in M$, we show that the graded vector space underlying the isotropy L_∞-algebra of $\mathcal{F}$ at p can be computed directly from $\mathcal{F}$. Moreover, for linear foliations, we show that the representation of the isotropy Lie algebra of $\mathcal{F}$ in p on the $(E_i)_p$ can also directly be computed from the foliation.

1.3.2 Chapter 3

The third chapter is the article "Stability of fixed points in Poisson geometry and higher Lie theory", and is available on the arXiv with identifier arXiv:2210.16256.

1.3.2.1 Main results

In this chapter, we discuss various aspects of the stability of fixed points under deformations of various structures. The question of stability of fixed points can be phrased as follows: given a geometric structure on a manifold M which induces a singular foliation on M, and a fixed point $p \in M$, when do all nearby geometric structures of the same type have a fixed point $q \in M$ near p? This is a special case of stability of leaves, as discussed in [CF10] for Lie algebroids and Poisson manifolds.

The article consists of three parts. In the first part, which goes up to and including Section 3.3.2, we give a proof of the stability result of [CF10, DW06] for fixed points of Lie algebroids, directly in terms of Lie algebroid data, while existing proofs used an identification of Lie algebroid structures with certain Poisson structures on the dual vector bundle. We then show that the both the problem of stability of fixed points and the condition to ensure a positive answer to the stability question can be formulated entirely in terms of the differential graded Lie algebra governing the deformations of Lie algebroid structures, serving as motivation for the main theorem of the article.

In the second part, which is Section 3.3, we state and prove the main theorem of the article (Theorem 3.3.20), which is an algebraic result about differential graded Lie algebras. We state a simplified version here, omitting technical assumptions.

Theorem. *Let $(\mathfrak{g}, \partial, [-,-])$ be a differential graded Lie algebra, and let $\mathfrak{h} \subseteq \mathfrak{g}$ be a differential graded Lie subalgebra of finite codimension. Let $Q \in \mathfrak{h}^1$ be a Maurer-Cartan element, and assume that*

$$H^1(\mathfrak{g}/\mathfrak{h}, \partial + [Q,-]) = 0.$$

Then for any Maurer-Cartan element $Q' \in \mathfrak{g}^1$ close enough to Q, there exists a smooth family $I \subseteq \mathfrak{g}^0$, parametrized by an open neighborhood of

$$\ker(\partial + [Q,-] : \mathfrak{g}^0/\mathfrak{h}^0 \to \mathfrak{g}^1/\mathfrak{h}^1),$$

such that for any $X \in I$, we have $(Q')^X \in \mathfrak{h}^1$.

The theorem therefore provides a sufficient condition for a positive answer to the stability question above Example 1.2.48.

In the third part, which is the remainder of the article, we give several applications of the main theorem, recovering in particular the known stability results for fixed points of Poisson structures and Lie algebroids ([CF10, DW06]), as well as the results for higher order fixed points of Poisson structures and Lie algebroids as in [DW06].

We then obtain new stability results for (higher order) fixed points of various geometric structures:

- Lie n-algebroids,
- Singular foliations, including a formal rigidity result,
- Lie bialgebroids,
- Poisson-Nijenhuis structures (including holomorphic Poisson structures),
- Courant algebroids,
- Dirac structures in a Courant algebroid of split signature that admit a complentary Dirac structure.

1.3.3 Chapter 4

The fourth chapter is the article "Stability of fixed points of Dirac structures", which is joint work with Marco Zambon, and is available on the arXiv with identifier arXiv:2304.12103.

1.3.3.1 Main results

This article consists of two parts. In the first part, up until and including Section 4.3, we generalize Theorem 3.3.20 as formulated in Section 1.3.2, to L_∞-algebras with a finite number of brackets, and an L_∞-subalgebra of degreewise finite codimension (Theorem 4.3.1). We state a simplified version here, omitting technical assumptions.

Theorem. *Let $(\mathfrak{g}, \{\ell_k\}_{1\leq k\leq n})$ be an $L_\infty[1]$-algebra, and let $\mathfrak{h} \subseteq \mathfrak{g}$ be an $L_\infty[1]$-subalgebra of degreewise finite codimension. Let $Q \in \mathfrak{h}^0$ be a Maurer-Cartan element. Assume that*

$$H^1(\mathfrak{g}/\mathfrak{h}, \overline{\ell_1^Q}) = 0.$$

Then for any Maurer-Cartan element $Q' \in \mathfrak{g}^1$ close enough to Q, there exists a smooth family $I \subseteq \mathfrak{g}^{-1}$, parametrized by an open neighborhood of

$$\ker(\ell_1^Q : \mathfrak{g}^{-1}/\mathfrak{h}^{-1} \to \mathfrak{g}^0/\mathfrak{h}^0),$$

such that for any $X \in I$, we have $(Q')^X \in \mathfrak{h}^1$.

This generalization is a natural improvement to 3.3.20, as the algebraic structure governing deformations of a geometric structure is often not a differential graded Lie algebra, but an L_∞-algebra. This can for instance be seen in the theory of deformations of Dirac structures [FZ15], regular foliations [Vit14], coisotropic submanifolds in Poisson geometry [OP05].

In the second part, which is the remainder of the article, we give an application to fixed points of Dirac structures to obtain Theorem 4.6.7, which improves Theorem 3.5.50, as it does not require the existence of a Dirac complement, and can be applied after simply picking a Lagrangian complement.

Theorem. *Let $(E, \langle -, - \rangle, \rho, [\![-, -]\!])$ be a Courant algebroid over M with split signature pairing. Let $A \subseteq E$ be a Dirac structure such that $p \in M$ is a zero-dimensional leaf, i.e. $\rho(A_p) = 0$. Let $\mathfrak{g} = A_p$ be the isotropy Lie algebra at p, and consider $\mathfrak{h} := \ker(\rho_p : E_p \to T_pM)^\perp \subseteq A_p$, where $\perp$ is taken with respect to $\langle -, - \rangle$. Assume that*

$$H^2\left(\frac{\wedge^\bullet \mathfrak{g}^*}{\wedge^\bullet \mathfrak{h}^\circ}, \overline{d_{\mathfrak{g}}}\right) = 0.$$

Then any Dirac structure near A has a smooth family of zero-dimensional leaves, locally parametrized by

$$\ker\left(\overline{d_A} : \frac{\mathfrak{g}^*}{\mathfrak{h}^\circ} \to \frac{\wedge^2 \mathfrak{g}^*}{\wedge^2 \mathfrak{h}^\circ}\right).$$

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Chapter 2

On the universal L_∞-algebroid of a linear foliation

This chapter contains the article [Sin22].

Abstract - We compute an L_∞-algebroid structure on a projective resolution of some classes of singular foliations on a vector space V induced by the linear action of some Lie subalgebras of $\mathfrak{gl}(V)$. This L_∞-algebroid provides invariants of the singular foliations, and also provides a constant-rank replacement of the singular foliation. The computation consists of first constructing a projective resolution of the foliation induced by the linear action of the Lie subalgebra $\mathfrak{g} \subseteq \mathfrak{gl}(V)$, and then computing the L_∞-algebroid structure. We then generalize these constructions to a vector bundle E, where the role of the origin is now taken by the zero section L.

We then show that the fibers over a fixed point of a projective resolution of any singular foliation can be computed directly from the foliation, without needing the projective resolution. For linear foliations, we also provide a way to compute the action of the isotropy Lie algebra in the origin on these fibers directly from the foliation.

2.1 Introduction

Let M be a smooth manifold, equipped with a singular foliation $\mathcal{F}$. By singular foliation, we mean a subsheaf $\mathcal{F}$ of the sheaf of vector fields $\mathfrak{X}$ on M such that

a) for all $U \subseteq M$ open, $\mathcal{F}(U)$ is a $C^\infty(U)$-submodule of $\mathfrak{X}(U)$,

b) for all $U \subseteq M$ open and $X, Y \in \mathcal{F}(U)$ we have $[X, Y] \in \mathcal{F}(U)$,

c) for all $x \in M$, there exists an open subset U_x of M containing x, such that $\mathcal{F}(U_x)$ is a finitely generated $C^\infty(U_x)$-module.

This definition of singular foliations was used in [Lav16, LGLS20]. An equivalent definition, using compactly supported vector fields, appeared in [Sus73, AS09] among other places. This equivalence was shown in [Wan17, Proposition 2.1.9], and the construction of the sheaf out of compactly supported vector fields appeared in [AZ16].
In [LGLS20], it was shown that under certain conditions on $\mathcal{F}$ one can associate an L_∞-algebroid over M to $(M, \mathcal{F})$. Here an L_∞-algebroid is a non-positively graded vector bundle $E = \bigoplus_{i \in \mathbb{Z}_{\leq 0}} E_i$, with a collection of multibrackets $\{\ell_k : \Gamma(\wedge^k E) \to \Gamma(E)\}_{k \geq 1}$, where ℓ_k has degree $2 - k$, and a vector bundle map $\rho : E_0 \to TM$ intertwining ℓ_2 with the Lie bracket of vector fields, called the anchor, satisfying some quadratic identities. L_∞-algebroids were first defined in [Vor10] as higher analogues of Lie algebroids. When $M = \{*\}$ is a single point, the definition reduces to that of a non-positively graded L_∞-algebra, which appeared in [LS93] as strongly homotopy Lie algebra. For the definition and important properties, we refer to [Lav22, Section 2.1].

The construction of [LGLS20] can be broken into two parts:

i) Choosing a resolution of $\mathcal{F}$ in the category of C^∞_M-modules by finitely generated projective modules[1],

ii) Constructing an L_∞-algebroid structure on the complex given by the resolution.

In step i) the conditions posed on $\mathcal{F}$ are used. Neither of the steps is constructive, but plenty of examples are given. Because of i), the L_∞-algebroid constructed in ii) satisfies a universality property, which implies uniqueness up to a notion of homotopy ([LGLS20, Corollary 2.9]). It will therefore be referred to as a *universal* L_∞-algebroid of $\mathcal{F}$. Because of the uniqueness up to homotopy, this L_∞-algebroid captures invariants of the singular foliation $\mathcal{F}$. Moreover, it allows to replace the singular foliation by a collection of constant-rank objects, which provides a framework to extend some results from the theory of Lie algebroids to singular foliations. Further, knowing a universal L_∞-algebroid of a singular

[1] As we work in the smooth category, this is equivalent to choosing a resolution of $\mathcal{F}(M)$ in the category of $C^\infty(M)$-modules by sections of vector bundles. We therefore do not distinguish between the sheaf and its global sections.

foliation allows to compute the modular class of a singular foliation as in [Lav22].

In this article, we generalize the example of the foliation $\mathcal{F}_0$ consisting of vector fields vanishing in the origin given in [LGLS20]. The foliation $\mathcal{F}_0$ is induced by the canonical linear $\mathfrak{gl}(V)$-action on V. The universal L_∞-algebroid given in [LGLS20, Example 3.99] only has two nonzero operations, turning it into a differential graded Lie algebroid (dg-Lie algebroid): $\ell_k = 0$ for $k \geq 3$. This raises several questions:

1) Can we construct the universal L_∞-algebroids for linear actions of other Lie algebras explicitly?
2) Can this approach be generalized to higher-dimensional leaves, with the corresponding isotropy Lie algebra?
3) Does the universal L_∞-algebroid for such a foliation always admit a dg-Lie algebroid structure (i.e. an L_∞-algebroid structure for which only the unary and binary brackets are non-zero)?

Main results

We address the questions above in the following examples:

- The Lie subalgebra $\mathfrak{gl}(V,W) \subseteq \mathfrak{gl}(V)$ for a given subspace $W \subseteq V$,
- the Lie subalgebra $\mathfrak{sl}(V) \subseteq \mathfrak{gl}(V)$ of traceless endomorphisms,
- the Lie subalgebra $\mathfrak{sp}(V,\omega) \subseteq \mathfrak{gl}(V)$ of endomorphisms preserving a non-degenerate skew-symmetric 2-form $\omega \in \wedge^2 V^*$.

All three questions have a positive answer in the cases $\mathfrak{gl}(V,W)$ and $\mathfrak{sl}(V)$. We answer questions 1) and 2) partially in the case of $\mathfrak{sp}(V,\omega)$, and we do not know the answer to question 3) in this case.
The resolutions of the module $\mathcal{F}$ we construct are *minimal* at the origin, which means that all differentials, being vector bundle maps, vanish at the origin. An advantage of this is that two L_∞-algebroid structures constructed on minimal resolutions are not only homotopy equivalent, but actually L_∞-isomorphic in a neighborhood of the origin, as explained at the end of Section 2.2.1.
In Section 2.2 we address questions 1) and 3).

- In Section 2.2.1 we recall the construction for $\mathfrak{gl}(V)$, which induces the foliation given by all vector fields vanishing in $0 \in V$, as given in Example 3.99 of [LGLS20].

- In Section 2.2.2 we consider the case of $\mathfrak{gl}(V,W)$, which induces the foliation generated by the linear vector fields tangent to the subspace W. We give a geometric resolution and describe an L_∞-algebroid structure with only a unary and binary bracket in Proposition 2.2.4 yielding a positive answer to question 3).

- In Section 2.2.3 we consider the case of $\mathfrak{sl}(V)$, which induces the foliation generated by linear vector fields preserving a constant volume form on V. We compute a geometric resolution in Proposition 2.2.6, and describe an L_∞-algebroid structure with only a unary and binary bracket in Proposition 2.2.8 yielding a positive answer to question 3).

- In Section 2.2.4 we fix a non-degenerate element $\omega \in \wedge^2 V^*$ and consider the case of $\mathfrak{sp}(V,\omega)$. We compute the geometric resolution in Proposition 2.2.12, and give a binary bracket in Proposition 2.2.15 depending on a map r^ω we chose. We show that this bracket does not satisfy the Jacobi identity, and give an expression for the ternary bracket. In the appendix 2.5.1 we investigate if the binary brackets can be simplified by picking r^ω to be a cochain map in some degrees, and show that this cannot be done when V is 4-dimensional. The answer to question 3) remains inconclusive in this case.

In Section 2.3 we turn our attention to the higher-dimensional analogues of the abovementioned cases and address the corresponding questions 2) and 3). In each of the cases the results of the earlier sections generalize.

- In Section 2.3.1 we consider the foliation of vector fields on a vector bundle E which are tangent to the zero section. We compute the geometric resolution in Proposition 2.3.5, and describe an L_∞-algebroid structure in Proposition 2.3.7.

- In Section 2.3.2 we consider the foliation of vector fields on a vector bundle which are tangent to a vector subbundle, of which the zero section is a special case. The geometric resolution and L_∞-algebroid structure are given in Proposition 2.3.8.

- In Section 2.3.3 we consider the foliation on an orientable vector bundle $E \to L$, with non-vanishing section $\mu \in \Gamma(\wedge^n E)$, where $n = \mathrm{rk}(E)$, generated (as C^∞_E-module) by the linear vector fields which preserve μ. We give the geometric resolution in Proposition 2.3.10, and the L_∞-algebroid structure in Proposition 2.3.11.

- In Section 2.3.4 we consider the foliation on a vector bundle $E \to L$ generated by the linear vector fields which preserve a non-degenerate

$\omega \in \Gamma(\wedge^2 E)$. The projective resolution is given in Proposition 2.3.12, and a binary bracket of the L_∞-algebroid structure in Proposition 2.3.13.

Finally, in Section 2.4 we consider a general foliation $\mathcal{F}$ on a vector space V, for which the origin p is a fixed point. We show that the fibers over p of any geometric resolution which is minimal at the origin can be computed directly from $\mathcal{F}$, *without needing to find a geometric resolution* (Proposition 2.4.2). In the case that $\mathcal{F}$ is linear, we additionally show that part of the structure of the isotropy L_∞-algebra (see [LGLS20, Section 4.2]), which is an invariant of the foliation $\mathcal{F}$, can be recovered from the foliation directly (Proposition 2.4.3).

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2.2 Zero-dimensional leaves

In this section, we compute a universal L_∞-algebroid for some classes of singular foliations generated by some Lie subalgebra of the Lie algebra of linear vector fields on a vector space V, addressing questions 1) and 3) from Section 2.1.

Convention. Throughout this section, for a finite-dimensional real vector space W, we will consider the trivial vector bundle $W \times V$ over a finite-dimensional real vector space V. Its global sections will be denoted by $\Gamma(W)$.
Unless stated otherwise, repeated indices will be summed over.

2.2.1 Vector fields vanishing at the origin

In this section we recall Example 3.99 of [LGLS20]. Let V be a real vector space of dimension $n \geq 0$, and let

$$\mathcal{F}_0(V) = \{X \in \mathfrak{X}(V) \mid X(0) = 0\} \tag{2.1}$$

be the submodule of vector fields on V vanishing in the origin. It is easy to see that it is a singular foliation. A resolution of $\mathcal{F}_0$ can be constructed using the following lemma.

Lemma 2.2.1. *The complexes*

$$0 \longrightarrow \Gamma(\wedge^n V^*) \xrightarrow{d_n} \dots \xrightarrow{d_2} \Gamma(V^*) \xrightarrow{\rho} I_q \longrightarrow 0 \ , \tag{2.2}$$

$$0 \longrightarrow \Gamma(\wedge^n V^*) \xrightarrow{d_n} \dots \xrightarrow{d_2} \Gamma(V^*) \xrightarrow{d_1} C^\infty(V) \xrightarrow{ev_q} \mathbb{R} \longrightarrow 0 \tag{2.3}$$

are exact. Here for $k = 1, \dots, n$, $d_k : \Gamma(\wedge^k V^*) \to \Gamma(\wedge^{k-1} V^*)$ and $\rho : \Gamma(V^*) \to I_q$ are the contraction with the Euler vector field $x^i \partial_{x^i}$, I_q is the ideal of functions vanishing at the origin $q \in V$ and ev_q is the evaluation of a function at $q = 0$. In particular, the complexes remain exact when applying the functor $- \otimes_{C^\infty(V)} \Gamma(W)$ for some vector bundle $W \times V \to V$.

Taking $W = V$ and tensoring (2.2) with $\Gamma(V) \cong \mathfrak{X}(V)$ we obtain the exact sequence

$$0 \longrightarrow \Gamma(\wedge^n V^* \otimes V) \xrightarrow{d_n} \dots \xrightarrow{d_2} \Gamma(V^* \otimes V) \xrightarrow{\rho} \mathcal{F}_0(V) \longrightarrow 0. \tag{2.4}$$

Here, and in the rest of this article we use the convention that $\Gamma(V^* \otimes V)$ sits in degree 0, and the differential $d_\bullet$ has degree 1.

An L_∞-algebroid structure on $\Gamma(\wedge^\bullet V^* \otimes V)$ can be given as follows: for the unary bracket, we take $d_\bullet$, as in (2.4). For the binary bracket we take the Nijenhuis-Richardson bracket: For $1 \leq k_1, k_2 \leq n$ define

$$[-,-] : \Gamma\left(\wedge^{k_1} V^* \otimes V\right) \times \Gamma\left(\wedge^{k_2} V^* \otimes V\right) \to \Gamma\left(\wedge^{k_1+k_2-1} V^* \otimes V\right)$$

by

$$\begin{aligned}[f_1 \cdot (\phi_1 \otimes w_1), f_2 \cdot (\phi_2 \otimes w_2)] &:= f_1 f_2 \cdot (\phi_1 \iota_{w_1}(\phi_2) \otimes w_2 - \epsilon_{k_1 k_2} \phi_2 \iota_{w_2}(\phi_1) \otimes w_1) \\ &+ \big(f_1 \rho(\phi_1 \otimes w_1)(f_2) \cdot (\phi_2 \otimes w_2) - f_2 \rho(\phi_2 \otimes w_2)(f_1) \cdot (\phi_1 \otimes w_1)\big)\end{aligned} \tag{2.5}$$

for $f_i \in C^\infty(V)$, $\phi_i \in \wedge^{k_i} V^*$, $w_i \in V$ $(i = 1, 2)$. Here $\epsilon_{k_1 k_2} = (-1)^{(k_1-1)(k_2-1)}$, for $v \in V, \alpha \in \wedge^k V^*$, $\iota_v(\alpha) \in \wedge^{k-1} V^*$ is the insertion of v into the first slot of α, and $\rho(\phi_i \otimes w_i)$ is understood to vanish if $k_i \neq 1$.
One can check that this defines a dg-Lie algebroid over V for which the image of ρ is exactly $\mathcal{F}_0$. We denote it by $L_\infty(\mathcal{F}_0)$.

Note that the differentials d_p vanish at the origin for $p = 2, \dots, n$. This implies that any L_∞-algebroid structure with the same property is L_∞-isomorphic to the one above in a neighborhood of the origin: by [LGLS20, Corollary 2.9], any two L_∞-algebroid structures are homotopy equivalent by an L_∞-morphism Φ.

By minimality and [LGLS20, Lemma 4.13iii)], this implies that the homotopy equivalence is an isomorphism in the origin. As invertibility is an open condition it follows that it is an isomorphism in a neighborhood of the origin.

2.2.2 Linear vector fields preserving a subspace

Let V be a real vector space of dimension n, and $W \subseteq V$ a linear subspace. Let

$$\mathcal{F}_W(V) := \{X \in \mathcal{F}_0(V) \mid X(I_W) \subseteq I_W\}$$

be the $C^\infty(V)$-submodule of linear vector fields tangent to the subspace W. This is a singular foliation, and is induced by the action of the Lie subalgebra $\mathfrak{gl}(V, W)$ of $\mathfrak{gl}(V)$ given by

$$\mathfrak{gl}(V, W) = \{A \in \mathfrak{gl}(V) \mid A(W) \subseteq W\},$$

the endomorphisms of V preserving W. The leaves of this foliation consist of the origin, the connected components of $W \setminus \{0\}$, and the connected components of $V \setminus W$.

Example 2.2.2. Let $V = \mathbb{R}^2, W = \{(x, 0) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$. Then $\mathcal{F}_W(V)$ is generated by the vector fields $x\partial_x, y\partial_x, y\partial_y$, and the leaves are the positive x-axis, the origin, the negative x-axis, the upper half plane and the lower half plane. In this case $\mathfrak{gl}(V, W)$ consists of all upper triangular matrices.

We can describe a minimal universal L_∞-algebroid of $\mathcal{F}_W$ as a L_∞-subalgebroid of $L_\infty(\mathcal{F}_0)$. In particular, it will again be a dg-Lie algebroid.

Definition 2.2.3. Let $j \in \{1, \ldots, n\}$. Define $K_j \subseteq \wedge^j V^* \otimes V = \mathrm{Hom}(\wedge^j V, V)$ by

$$K_j := \{\phi \in \wedge^j V^* \otimes V \mid \forall w \in W, \forall v_1, \ldots, v_{j-1} \in V : \phi(w, v_1, \ldots, v_{j-1}) \in W\}.$$

Proposition 2.2.4.

i) The differential

$$d_j : \Gamma(\wedge^j V^* \otimes V) \to \Gamma(\wedge^{j-1} V^* \otimes V)$$

as in (2.4) *restricts to a map*

$$d_j : \Gamma(K_j) \to \Gamma(K_{j-1}).$$

ii) The bracket (2.5) *restricts to the subspaces* $\Gamma(K_j)$.

iii) The subcomplex

$$0 \longrightarrow \Gamma(K_n) \xrightarrow{d_n} \dots \xrightarrow{d_2} \Gamma(K_1) \xrightarrow{\rho_W} \mathcal{F}_W(V) \longrightarrow 0 \quad (2.6)$$

is exact, where $\rho_W = \rho|_{\Gamma(K_1)}$.

Consequently, $\Gamma(K_\bullet)$ *with the restrictions of* $d_\bullet$ *and* $[-,-]$ *is a minimal universal* L_∞*-algebroid of the foliation* $\mathcal{F}_W$.

Proof. Items i) and ii) are straightforward computations. For item iii), fix a complement C of W in V. Then K_i can be identified with $\wedge^i V^* \otimes W \oplus \wedge^i C^* \otimes C$, and the complex (2.6) decomposes as

$$(\Gamma(K_\bullet), \partial) = (\Gamma(\wedge^\bullet V^* \otimes W) \oplus \Gamma(\wedge^\bullet C^* \otimes C), \partial_W + \partial_C),$$

where

$$\partial_W(\phi) = x^i \iota_{e_i}(\phi)$$

for $\phi \in \Gamma(\wedge^i V^* \otimes W)$, and $\{e_i\}_{i=1}^n$ is a basis for V, with linear coordinates $\{x^i\}_{i=1}^n$. For $\psi \in \Gamma(\wedge^i C^* \otimes C)$,

$$\partial_C(\psi) = y^i \iota_{f_i}(\psi),$$

where $\{f_i\}_{i=1}^r$ is a basis for C, and $\{y^i\}_{i=1}^r$ are the corresponding linear coordinates. By Lemma 2.2.1, both are exact, concluding the proof. □

2.2.3 Vector fields preserving a volume form

The next choice for a Lie algebra $\mathfrak{g}$ acting linearly on a vector space V we consider is $\mathfrak{g} = \mathfrak{sl}(V)$, the Lie algebra of traceless endomorphisms. Observe that the partition of V is identical to the case of $\mathfrak{gl}(V)$, but that the underlying submodules of $\mathfrak{X}_V$ are different. Let $\mu \in \wedge^n V^*$ be a non-zero element, and denote the foliation given by the action of $\mathfrak{sl}(V)$ by $\mathcal{F}_\mu$.

2.2.3.1 The projective resolution

As it is in general not possible to restrict a projective resolution of a module to a submodule, one cannot directly get a projective resolution of the module of vector fields generated by the action of $\mathfrak{sl}(V)$, by restricting all modules to live over $\mathfrak{sl}(V)$. But for the most part, the resolution we will construct is related to the one given in (2.4). Consider the following diagram of $C^\infty(V)$-modules:

$$\begin{array}{ccc} \Gamma(\wedge^2 V^* \otimes V) & \xrightarrow{d_2} & \Gamma(V^* \otimes V) \\ & & \big\downarrow \mathrm{Tr} \\ \Gamma(V^*) & \xrightarrow{\partial_1} & \Gamma(\mathbb{R}) \end{array} , \tag{2.7}$$

where $\partial_1 : \Gamma(V^*) \to \Gamma(\mathbb{R})$ is the contraction with the negative of the Euler vector field $x^i \partial_{x^i}$ and Tr is the trace of endomorphisms.
Now there is a linear map

$$\phi_2 : \wedge^2 V^* \otimes V \to V^*$$

by taking partial traces: for $\psi \in \wedge^2 V^*, v \in V$, we set

$$\phi_2(\psi \otimes v) = -\iota_v(\psi).$$

We now claim that (the constant extension of) ϕ_2 completes (2.7) to an anti-commutative square. Indeed: let $\{e_i\}_{i=1}^n$ be a basis of V, with corresponding coordinates $\{x^i\}_{i=1}^n$, and let $\psi \otimes v \in \Gamma(\wedge^2 V^* \otimes V)$. Then

$$\begin{aligned} \partial_1(\phi_2(\psi \otimes v)) &= -\partial_1(\iota_v(\psi)) \\ &= x^i \iota_{e_i} \iota_v(\psi) \\ &= -x^i \iota_v \iota_{e_i}(\psi) \\ &= -\mathrm{Tr}(x^i \iota_{e_i}(\psi) \otimes v) \\ &= -\mathrm{Tr}(d_2(\psi \otimes v)). \end{aligned}$$

More generally, for $1 \leq k \leq n$, we can define the anti-symmetrized partial trace map

$$\phi_k : \wedge^k V^* \otimes V \to \wedge^{k-1} V^*.$$

For $\alpha \in \wedge^k V^*, v \in V$, we set

$$\phi_k(\alpha \otimes v) = (-1)^{k-1} \iota_v(\alpha).$$

Observe that ϕ_1 is the usual trace.
Note that the map $\partial_1 : \Gamma(V^*) \to \Gamma(\mathbb{R}) = C^\infty(V)$ as in (2.7) of free $C^\infty(V)$-modules can be extended to obtain a cochain complex

$$0 \longrightarrow \Gamma(\wedge^n V^*) \xrightarrow{\partial_n} \dots \xrightarrow{\partial_2} \Gamma(V^*) \xrightarrow{\partial_1} C^\infty(V) \longrightarrow 0 . \tag{2.8}$$

The cochain complex is concentrated in negative degrees, with $C^\infty(V)$ being in degree -1. Note that by Lemma 2.2.1, the complex is exact in degrees $-2, \ldots, -n-1$, as it is the truncation of (2.3). The following lemma describes the compatibility of ϕ with the respective differentials:

Lemma 2.2.5. *The map* $\phi : (\Gamma(\wedge^\bullet V^* \otimes V), d_\bullet) \to (\Gamma(\wedge^{\bullet-1} V^*), \partial_\bullet)$ *is a cochain map of degree* -1*, which is surjective in degrees* $0, \ldots, -(n-2)$*, and an isomorphism in degree* $-n+1$*, where* $d_\bullet$ *is as in* (2.4)*, and* $\partial_\bullet$ *is as in* (2.8)*.*

Proof. Let $\alpha \otimes v \in \Gamma(\wedge^{k+1} V^* \otimes V)$. We first show that ϕ anti-commutes with the respective differential:

$$\begin{aligned}
\partial_k(\phi_{k+1}(\alpha \otimes v)) &= \partial_k((-1)^k \iota_v(\alpha)) \\
&= (-1)^{k+1} x^i \iota_{e_i}(\iota_v(\alpha)) \\
&= (-1)^k x^i \iota_v(\iota_{e_i}(\alpha)) \\
&= -\phi_k(x^i \iota_{e_i}(\alpha) \otimes v) \\
&= -\phi_k(d_{k+1}(\alpha \otimes v)).
\end{aligned}$$

To see the surjectivity, pick a basis $\{e_i\}_{i=1}^n$ of V, and a dual basis $\{e^i\}_{i=1}^n$ of V^* such that $\mu = e^1 \wedge \cdots \wedge e^n$. For $k \in \{1, \ldots, n\}$, a basis for $\wedge^{k-1} V^*$ is given by $\{e^{i_1} \wedge \cdots \wedge e^{i_{k-1}} \mid 1 \leq i_1 < \cdots < i_{k-1} \leq n\}$. Given $e^{i_1} \wedge \cdots \wedge e^{i_{k-1}}$, let $q \in \{1, \ldots, n\} - \{i_1, \ldots, i_{k-1}\}$. Then

$$\phi_k(e^{i_1} \wedge \cdots \wedge e^{i_{k-1}} \wedge e^q \otimes e_q) = e^{i_1} \wedge \cdots \wedge e^{i_{k-1}},$$

where q is *not* summed over. Further, under the identification

$$\begin{aligned}
V &\to \wedge^n V^* \otimes V \\
v &\mapsto e^1 \wedge \cdots \wedge e^n \otimes v,
\end{aligned}$$

ϕ_n is the map $V \to \wedge^{n-1} V^*$ given by contraction with the volume form $e^1 \wedge \cdots \wedge e^n$, which is an isomorphism. □

We use the properties of ϕ to construct a projective resolution for $\mathcal{F}_\mu$.

Proposition 2.2.6. *For* $i = 1, \ldots, n$*, let* $K_i \subseteq \wedge^i V^* \otimes V$ *be defined by*

$$K_i := \ker(\phi_i).$$

The sequence

$$0 \longrightarrow \Gamma(\wedge^n V^*) \xrightarrow{d_n \phi_n^{-1} \partial_n} \Gamma(K_{n-1}) \xrightarrow{d_{n-1}} \dots \xrightarrow{d_2} \Gamma(K_1) \xrightarrow{\rho_\mu} \mathcal{F}_\mu(V) \longrightarrow 0 \tag{2.9}$$

is exact, where $\rho_\mu = \rho|_{\Gamma(K_1)}$.

Proof. Note that by definition of ϕ_1, $K_1 = \mathfrak{sl}(V)$, so ρ_μ is surjective by definition of $\mathcal{F}_\mu(V)$.

Let $i \in \{1, \dots, n-2\}$. Consider the following diagram with (anti)-commuting squares, where the middle and bottom rows are exact by Lemma 2.2.1:

$$\begin{array}{ccccccc}
\Gamma(K_{i+2}) & \xrightarrow{d_{i+2}} & \Gamma(K_{i+1}) & \xrightarrow{d_{i+1}} & \Gamma(K_i) & \xrightarrow{d_i} & \Gamma(K_{i-1}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Gamma(\wedge^{i+2} V^* \otimes V) & \xrightarrow{d_{i+2}} & \Gamma(\wedge^{i+1} V^* \otimes V) & \xrightarrow{d_{i+1}} & \Gamma(\wedge^{i} V^* \otimes V) & \xrightarrow{d_i} & \Gamma(\wedge^{i-1} V^* \otimes V) \\
\downarrow \phi_{i+2} & & \downarrow \phi_{i+1} & & \downarrow \phi_i & & \downarrow \phi_{i-1} \\
\Gamma(\wedge^{i+1} V^*) & \xrightarrow{\partial_{i+1}} & \Gamma(\wedge^{i} V^*) & \xrightarrow{\partial_i} & \Gamma(\wedge^{i-1} V^*) & \xrightarrow{\partial_{i-1}} & \Gamma(\wedge^{i-2} V^*)
\end{array}$$

For exactness at $\Gamma(K_i)$, take $\chi \in \Gamma(K_i)$ such that $d_i(\chi) = 0$, where d_1 is understood to be ρ_μ. Then by exactness of the middle row, there exists $\psi \in \Gamma(\wedge^{i+1} V^* \otimes V)$ such that $d_{i+1}(\psi) = \chi$. Now ψ may not be an element of $\Gamma(K_{i+1})$, so we consider $\phi_{i+1}(\psi)$.

Note that

$$\partial_i \phi_{i+1}(\psi) = -\phi_i(\partial_{i+1}(\psi)) = -\phi_i(\chi) = 0,$$

so by exactness of (2.8) there exists $\tau \in \Gamma(\wedge^{i+1} V^*)$ such that

$$\phi_{i+1}(\psi) = \partial_{i+1}(\tau).$$

Using surjectivity of ϕ_{i+2}, lift τ to an element $\tilde{\tau} \in \Gamma(\wedge^{i+2} V^* \otimes V)$. Then

$$\phi_{i+1}(\psi + \partial_{i+2}(\tilde{\tau})) = \phi_{i+1}(\psi) - \partial_{i+1}(\tau) = 0,$$

so $\psi + \partial_{i+2}(\tilde{\tau}) \in K_{i+1}$, and

$$\partial_{i+1}(\psi + \partial_{i+2}(\tilde{\tau})) = \chi.$$

For exactness at $\Gamma(K_{n-1})$, consider

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & \Gamma(\wedge^n V^*) & \xrightarrow{d_n (\phi_n)^{-1} \partial_n} & \Gamma(K_{n-1}) & \xrightarrow{d_{n-1}} & \Gamma(K_{n-2}) \\
 & & & & & & \downarrow & & \downarrow \\
 & & 0 & \longrightarrow & \Gamma(\wedge^n V^* \otimes V) & \xrightarrow{d_n} & \Gamma(\wedge^{n-1} V^* \otimes V) & \xrightarrow{d_{n-1}} & \Gamma(\wedge^{n-2} V^* \otimes V) \\
 & & & & \downarrow \phi_n & & \downarrow \phi_{n-1} & & \downarrow \phi_{n-2} \\
\Gamma(\wedge^n V^*) & \xrightarrow{\partial_n} & \Gamma(\wedge^{n-1} V^*) & & \xrightarrow{\partial_{n-1}} & & \Gamma(\wedge^{n-2} V^*) & \xrightarrow{\partial_{n-2}} & \Gamma(\wedge^{n-3} V^*)
\end{array} .$$

Let $\xi \in \Gamma(K_{n-1})$ such that
$$d_{n-1}(\xi) = 0.$$
Then there exists $\eta \in \Gamma(\wedge^n V^* \otimes V)$ such that
$$d_n(\eta) = \xi.$$
As ϕ_n is an isomorphism, we have $\eta = \phi_n^{-1}(\phi_n(\eta))$. Moreover, we know that
$$\partial_{n-1}(\phi_n(\eta)) = -\phi_{n-1}(d_n(\eta)) = -\phi_{n-1}(\xi) = 0,$$
so
$$\phi_n(\eta) = \partial_n(\pi)$$
for some $\pi \in \Gamma(\wedge^n V^*)$. Consequently,
$$\xi = d_n(\phi_n^{-1}(\partial_n(\pi))).$$
Finally, exactness at $\Gamma(\wedge^n V^*)$ is clear. □

2.2.3.2 The L_∞-algebroid structure

In this section, we will construct the L_∞-algebroid structure on the resolution (2.9) of $\mathcal{F}_\mu(V)$. As in most degrees the spaces involved in the resolution of $\mathcal{F}_\mu$ are contained in the spaces involved in the resolution of $\mathcal{F}_0$, we try to use the restriction of (2.5). The following lemma shows that this can be done:

Lemma 2.2.7. *The bracket* (2.5) *restricts to the subspaces* $\Gamma(K_i)$*.*

This gives us a hint on how to extend the bracket to (2.9): on the subcomplex given by the part up until degree $n-1$, it is given by (2.5). Note that there is no issue when $k_1 + k_2 - 1 = n$: since the bracket should land in $\Gamma(K_n) = 0$, we can unambiguously extend this definition when we replace $\Gamma(K_n)$ by $\Gamma(\wedge^n V^*)$. For degree reasons and the Leibniz identity in a L_∞-algebroid, we only have to specify what happens when we pair the constant extension of $X \in K_1 = \mathfrak{sl}(V)$ with the constant extension of $\mu \in \wedge^n V^*$. Due to the requirement that the differential is a derivation of the binary bracket, there is only one choice for this: We set
$$[X, \mu] := 0 \in \Gamma(\wedge^n V^*). \tag{2.10}$$
We then obtain:

Proposition 2.2.8. *The binary operation defined by the restriction of* (2.5) *on the spaces* $\Gamma(K_i)$*, together with the extension of* (2.10) *defines a dg-Lie algebroid structure on the resolution* (2.9) *of* $\mathcal{F}_\mu(V)$*. This is a universal* L_∞*-algebroid of* $\mathcal{F}_\mu$*, which is minimal at the origin.*

2.2.4 Vector fields preserving the linear symplectic form

Next, we consider the symplectic Lie algebra. Given a vector space V of even dimension n, and a non-degenerate skew-symmetric bilinear map $\omega : V \times V \to \mathbb{R}$, we consider the Lie subalgebra of $\mathfrak{gl}(V)$ preserving ω:

Definition 2.2.9. Let (V, ω) be a symplectic vector space. The *symplectic Lie algebra* is the Lie subalgebra of $\mathfrak{gl}(V)$ given by

$$\mathfrak{sp}(V, \omega) := \{A \in \mathfrak{gl}(V) \mid \omega(Ax, y) + \omega(x, Ay) = 0 \ \ \forall x, y \in V\}, \tag{2.11}$$

By restricting the anchor ρ to $\Gamma(\mathfrak{sp}(V, \omega))$, we obtain a singular foliation

$$\mathcal{F}_\omega(V) := \rho(\Gamma(\mathfrak{sp}(V, \omega)).$$

In Section 2.2.4.1 we construct a projective resolution of $\mathcal{F}_\omega(V)$ (Proposition 2.2.12). In Section 2.2.4.2 we construct a part of the L_∞-algebroid structure. We give an expression for the binary bracket (Proposition 2.2.15), depending on a choice of left inverse r^ω of an injective cochain map. This bracket does not satisfy the Jacobi identity, so we give an expression for the ternary bracket, which serves as a contracting homotopy for the Jacobiator. In appendix 2.5.1 we investigate whether r^ω can be chosen to be a cochain map in some degrees, which would simplify the binary bracket. We show that when $\dim(V) = 4$, r^ω can not be chosen as a cochain map in any degree (Proposition 2.5.5).

2.2.4.1 The projective resolution

As before, we first construct the projective resolution of the foliation $\mathcal{F}_\omega = \rho(\Gamma(\mathfrak{sp}(V, \omega)))$ on V. The starting point is the same as for $\mathfrak{sl}(V)$, but the rest of the approach will be quite different, as the analog of the map ϕ is not surjective in negative degrees. First consider the map $\phi_1^\omega : \mathfrak{gl}(V) \to \wedge^2 V^*$ given by

$$\phi_1^\omega(A) = A \cdot \omega$$

for $A \in \mathfrak{gl}(V)$, where for $x, y \in V$,

$$(A \cdot \omega)(x, y) = \omega(Ax, y) + \omega(x, Ay).$$

The next step is to extend this map to the entire dg-Lie algebra $\Gamma(\wedge^\bullet V^* \otimes V)$, as in (2.4) and (2.5). This immediately raises the question what the codomain should be. We define for $p = 1, \ldots, n$

$$\phi_p^\omega : \wedge^p V^* \otimes V \to \wedge^{p-1} V^* \otimes \wedge^2 V^*$$

by

$$\phi^\omega_p(\alpha \otimes X) := (-1)^{p-1}\iota_{e_i}(\alpha) \otimes e^i \wedge \iota_X\omega,$$

where $\{e_i\}_{i=1}^n$ and $\{e^i\}_{i=1}^n$ are dual bases of V and V^* respectively. When viewing the domain and codomain of ϕ^ω_p as $\mathrm{Hom}(\wedge^p V, V)$ and $\mathrm{Hom}(\wedge^{p-1}V, \wedge^2 V^*)$ respectively, the map ϕ^ω_p can equivalently be described as

$$\phi^\omega_p(\psi)(v_1, \ldots, v_{p-1}) = \psi(v_1, \ldots, v_{p-1}, -) \cdot \omega$$

for $\psi \in \mathrm{Hom}(\wedge^p V, V)$, $v_1, \ldots, v_{p-1} \in V$.

We now equip the graded $C^\infty(V)$-module $\Gamma(\wedge^\bullet V^* \otimes \wedge^2 V^*)$ with the differential

$$\partial_p : \Gamma(\wedge^p V^* \otimes \wedge^2 V^*) \to \Gamma(\wedge^{p-1} V^* \otimes \wedge^2 V^*)$$

given by

$$\partial_p(\alpha \otimes \tau) := -x^i \iota_{e_i}(\alpha) \otimes \tau.$$

Here the grading is chosen as

$$\Gamma(\wedge^\bullet V^* \otimes \wedge^2 V^*)^p = \Gamma(\wedge^{p+1} V^* \otimes \wedge^2 V^*).$$

Finally, there is an action of the dg-Lie algebra $\Gamma(\wedge^\bullet V^* \otimes V)$ on $\Gamma(\wedge^\bullet V^* \otimes \wedge^2 V^*)$: for $\alpha \otimes X \in \wedge^p V^* \otimes V, \beta \otimes \tau \in \wedge^q V^* \otimes \wedge^2 V^*$, we set

$$(\alpha \otimes X) \cdot (\beta \otimes \tau) := (-1)^{p-1}\alpha \iota_X(\beta) \otimes \tau + \iota_{e_i}(\alpha)\beta \otimes e^i \wedge \iota_X(\tau) \tag{2.12}$$

for constant sections, and extend it to non constant sections using the Leibniz rule with respect to the anchor of $\Gamma(\mathfrak{gl}(V))$. Note that by multiplying by $-(-1)^{(p-1)(q-1)}$, we can turn this into a right action. We have the following lemma summarizing the properties of the data above, of which the proof is a direct computation.

Lemma 2.2.10.

i) ϕ^ω_1 *is surjective, and* $\ker(\phi^\omega_1) = \mathfrak{sp}(V, \omega)$.

ii) For $k \geq 2$, ϕ^ω_k *is injective. In particular,* ϕ^ω_2 *is an isomorphism.*

iii) ϕ^ω *is a cochain map of degree* -1, *i.e. we have*

$$\phi^\omega_p d_{p+1} + \partial_p \phi^\omega_{p+1} = 0.$$

iv) The operation defined in (2.12) *is a dg-Lie algebra action. Consequently, there is a dg-Lie algebra structure on* $\Gamma(\wedge^\bullet V^* \otimes V) \oplus \Gamma(\wedge^\bullet V^* \otimes \wedge^2 V^*)$ *encoding this action.*
For $p \geq 0$, *the degree* p*-part is given by* $\Gamma(\wedge^{p+1} V^* \otimes V) \oplus \Gamma(\wedge^{p+1} V^* \otimes \wedge^2 V^*)$, *and the differential is given by* $d + \partial$.

v) ϕ^ω *is a derivation of the bracket on* $\Gamma(\wedge^\bullet V^* \otimes V) \oplus \Gamma(\wedge^\bullet V^* \otimes \wedge^2 V^*)$*: for* $\alpha \otimes X \in \Gamma(\wedge^p V^* \otimes V), \beta \otimes Y \in \Gamma(\wedge^q V^* \otimes V)$*, we have the equality*

$$\phi^\omega_{p+q-1}([\alpha\otimes X, \beta\otimes Y]) = [\phi^\omega_p(\alpha\otimes X), \beta\otimes Y] + (-1)^{p-1}[\alpha\otimes X, \phi^\omega_q(\beta\otimes Y)]. \tag{2.13}$$

Corollary 2.2.11. *By property iii), the differentials* $d_\bullet$ *and* $\partial_\bullet$ *restrict and descend to the kernel and cokernel of* ϕ^ω *respectively. We denote the differential induced by* $\partial_\bullet$ *on* $\Gamma(\mathrm{coker}(\phi^\omega))$ *by* $\bar{\partial}_\bullet$*.*

Using these properties, we can construct a projective resolution:

Proposition 2.2.12. *For* $i = 3, \ldots, n+1$*, let* C_i *be defined as* $C_i := \mathrm{coker}(\phi^\omega_i)$*. The sequence*

$$0 \longrightarrow \Gamma(C_{n+1}) \xrightarrow{\bar{\partial}_n} \ldots \xrightarrow{\bar{\partial}_3} \Gamma(C_3) \xrightarrow{d_2(\phi^\omega_2)^{-1}\partial_2} \Gamma(K) \longrightarrow \mathcal{F}_\omega(V) \longrightarrow 0\ , \tag{2.14}$$

is exact, where $K = \mathfrak{sp}(V,\omega)$*. Here* $\phi^\omega_{n+1} : 0 \to \Gamma(\wedge^n V^* \otimes \wedge^2 V^*)$ *is understood to be the zero map.*

Proof. We start by proving exactness at $\Gamma(C_p)$ for $p = 4, \ldots, n+1$. Consider the diagram

$$\begin{array}{ccccccc}
\ldots \xrightarrow{d_{p+1}} & \Gamma(\wedge^p V^* \otimes V) & \xrightarrow{d_p} & \Gamma(\wedge^{p-1} V^* \otimes V) & \xrightarrow{d_{p-1}} & \Gamma(\wedge^{p-2} V^* \otimes V) \\
& \downarrow{\scriptstyle \phi^\omega_p} & & \downarrow{\scriptstyle \phi^\omega_{p-1}} & & \downarrow{\scriptstyle \phi^\omega_{p-2}} \\
\ldots \xrightarrow{\partial_p} & \Gamma(\wedge^{p-1} V^* \otimes \wedge^2 V^*) & \xrightarrow{\partial_{p-1}} & \Gamma(\wedge^{p-2} V^* \otimes \wedge^2 V^*) & \xrightarrow{\partial_{p-2}} & \Gamma(\wedge^{p-3} V^* \otimes \wedge^2 V^*) \\
& \downarrow & & \downarrow & & \downarrow \\
\ldots \xrightarrow{\overline{\partial_p}} & \Gamma(C_p) & \xrightarrow{\overline{\partial_{p-1}}} & \Gamma(C_{p-1}) & \xrightarrow{\overline{\partial_{p-2}}} & \Gamma(C_{p-2})
\end{array}$$

For $\tau \in \Gamma(\wedge^{p-1} V^* \otimes \wedge^2 V^*)$, assume that there exists $X \in \Gamma(\wedge^{p-1} V^* \otimes V)$ such that

$$\partial_{p-1}(\tau) = \phi^\omega_{p-1}(X).$$

Then

$$\phi^\omega_{p-2}(d_{p-1}(X)) = -\partial_{p-2}(\phi^\omega_{p-1}(X)) = 0,$$

and by injectivity of ϕ^ω_{p-1}, it follows that $d_{p-1}(X) = 0$. By exactness of (2.4) we find $X = d_p(Y)$. Now

$$\partial_{p-1}(\tau + \phi^\omega_p(Y)) = \phi^\omega_{p-1}(X) - \phi^\omega_{p-1}(X) = 0,$$

so by exactness of (2.3), with $W = \wedge^2 V^*$, we find that there exists $\mu \in \Gamma(\wedge^p V^* \otimes \wedge^2 V^*)$ such that

$$\tau = \partial_p(\mu) - \phi_p^\omega(Y),$$

showing that the class of τ modulo the image of ϕ^ω is a coboundary.

For exactness at $\Gamma(C_3)$, assume that for $\tau \in \Gamma(\wedge^2 V^* \otimes \wedge^2 V^*)$ such that

$$d_2(\phi_2^\omega)^{-1}\partial_2(\tau) = 0.$$

Then there exists $X \in \Gamma(\wedge^3 V^* \otimes V)$ with

$$(\phi_2^\omega)^{-1}\partial_2(\tau) = d_3(X),$$

or

$$\partial_2(\tau + \phi_2^\omega(X)) = 0,$$

which implies that there exists $\mu \in \Gamma(\wedge^3 V^* \otimes \wedge^2 V^*)$ such that

$$\partial_3(\mu) = \tau + \phi_2^\omega(X),$$

so the class of τ module the image of ϕ^ω is a coboundary.
Finally, by definition of $\mathcal{F}_\omega$ the anchor restricted to $\mathfrak{sp}(V,\omega)$ is surjective, so it suffices to show that the kernel is precisely $(d_2(\phi_2^\omega)^{-1}\partial_2)(\Gamma(\wedge^2 V^* \wedge^2 V^*)$. Let $A \in \Gamma(\mathfrak{sp}(V,\omega))$ such that

$$\rho_\omega(A) = 0.$$

Then $A = d_2(X)$ for some $X \in \Gamma(\wedge^2 V^* \otimes V)$. As ϕ_2^ω is an isomorphism, we have

$$A = d_2(\phi_2^\omega)^{-1}\phi_2^\omega(X).$$

As

$$\partial_1\phi_2^\omega(X) = -\phi_1^\omega(d_2(X)) = \phi_1^\omega(A) = 0,$$

it follows that $\phi_2^\omega(X) = \partial(\mu)$ for some $\mu \in \Gamma(\wedge^2 V^* \otimes \wedge^2 V^*)$, and

$$A = d_2(\phi_2^\omega)^{-1}\partial_2(\mu),$$

concluding the proof. □

2.2.4.2 The (partial) L_∞-algebroid structure

In this section we construct part of an L_∞-algebroid structure on the resolution (2.14). Due to the algebraic structures present, there is a canonical Lie bracket

$\{-,-\}$ on the graded $C^\infty(V)$-module underlying the resolution (2.14). It is given by

$$\{A, B\} = [A, B]$$

for $A, B \in \Gamma(\mathfrak{sp}(V, \omega))$, where $[-,-]$ is the usual bracket, and

$$\{A, \omega_k + \mathrm{im}(\phi^\omega_{k+1})\} = [A, \omega_k] + \mathrm{im}(\phi^\omega_{k+1})$$

for $A \in \Gamma(\mathfrak{sp}(V, \omega)), \omega_k \in \Gamma(\mathrm{coker}(\phi^\omega_{k+1}))$. Here the bracket $[-,-]$ on the right hand side is the semi-direct product bracket as described in Lemma 2.2.10iv).

Remark 2.2.13. One way to see that $\{-,-\}$ is well-defined, is by forgetting the differentials $d_\bullet$ and $\partial_\bullet$, and viewing $(\Gamma(\wedge^\bullet V^*) \oplus \Gamma(\wedge^\bullet V^* \otimes \wedge^2 V^*), \phi^\omega)$ as a dg-Lie algebra. The resolution (2.14) of $\mathcal{F}_\omega(V)$ is then precisely the cohomology of this dg-Lie algebra. Consequently, the bracket equips the graded module with a graded Lie algebra structure.

When at least one of the entries of $\{-,-\}$ has degree 0, the differential in (2.14) is a derivation of $\{-,-\}$. However, for elements $\omega_1, \omega_2 \in \Gamma(\mathrm{coker}(\phi^\omega_3))$, for the differential in (2.14) to be a derivation of the bracket, the equation

$$[d_2(\phi^\omega_2)^{-1}\partial_2\omega_1, \omega_2] - [\omega_1, d_2(\phi^\omega_2)^{-1}\partial_2\omega_2] = \overline{\partial_3}[\omega_1, \omega_2] = 0 \in \Gamma(C_4) \qquad (2.15)$$

must hold. This means that the expression (2.15) must lie in the image of ϕ^ω_4, but one can check that this is not the case: the binary operation $\{-.-\}$ therefore does not equip the resolution (2.14) with a L_∞-algebroid structure, as the differential is not a derivation of the binary bracket.
To rectify this, we modify $\{-.-\}$ to obtain a new binary operation $[\![-,-]\!]$ on the resolution (2.14), for which the differential is a derivation.
Before we define this binary operation, we make a choice of left inverse of the map

$$\phi^\omega_p : \Gamma(\wedge^p V^* \otimes V) \to \Gamma(\wedge^{p-1} V^* \otimes \wedge^2 V^*)$$

for $p \geq 2$. Define

$$r^\omega_p : \wedge^p V^* \otimes \wedge^2 V^* \to \wedge^{p+1} V^* \otimes V$$

by

$$r^\omega_p(\omega_p \otimes \tau) = \left(\frac{1}{p+1}\omega_p \wedge \iota_{e_i}(\tau) - \frac{(-1)^p}{p(p+1)}\iota_{e_i}(\omega_p) \wedge \tau\right) \otimes \omega^{-1}(e^i) \in \wedge^{p+1} V^* \otimes V$$

for $\omega_p \in \wedge^p V^*, \tau \in \wedge^2 V^*$. Further, $\{e_i\}^n_{i=1}, \{e^i\}^n_{i=1}$ are dual bases for V and V^* respectively, and $\omega^{-1} : V^* \to V$ is the inverse of the contraction map $\omega : V \to V^*$. The proof of the following lemma is a straightforward computation:

Lemma 2.2.14.

i) For $k \geq 2$, r_k^ω intertwines the $\mathfrak{sp}(V,\omega)$-action on $\Gamma(\wedge^k V^* \otimes \wedge^2 V^*)$ and $\Gamma(\wedge^{k+1} V^* \otimes V)$.

ii) For $k \geq 2$,

$$r_k^\omega \circ \phi_{k+1}^\omega = id_{\Gamma(\wedge^{k+1}V^*\otimes V)}.$$

We can now give an expression for the binary operation $[\![-,-]\!]$ for which the differential of the resolution (2.14) is a derivation, providing the binary bracket for an L_∞-algebroid structure on the resolution.

Proposition 2.2.15. *When at least one entry of $[\![-,-]\!]$ has degree 0, we set*

$$[\![-,-]\!] = \{-,-\}.$$

Now let $p, q \geq 2$. For $\omega_p \in \Gamma(\wedge^p V^* \otimes \wedge^2 V^*), \omega_q \in \Gamma(\wedge^q V^* \otimes \wedge^2 V^*)$, set

$$[\![\omega_p, \omega_q]\!] := [r_{p-1}^\omega \partial_p P_p(\omega_p), P_q(\omega_q)] + [P_p(\omega_p), r_{q-1}^\omega \partial_q P_q(\omega_q)] \mod im(\phi_{p+q}^\omega) \tag{2.16}$$

in

$$\Gamma(\text{coker}(\phi_{p+q}^\omega))$$

.Here $P_p : \Gamma(\wedge^p V^* \otimes \wedge^2 V^*) \to \Gamma(\ker(r_p^\omega))$ is the projection

$$P_p = id - \phi_{p+1}^\omega \circ r_p^\omega,$$

and $[-,-]$ on the right hand side is the semi-direct product bracket as described in Lemma 2.2.10iv).
Then the differential of (2.14) is a derivation of $[\![-,-]\!]$.

Proof. The proof is a direct computation using Lemma 2.2.10iv) and Lemma 2.2.14ii). □

The natural question is now: Does $[\![-,-]\!]$ satisfy the Jacobi identity?

To address this, we distinguish two cases. We first compute the Jacobiator when at least one of the entries has degree 0, and then when all the entries have negative degree.

- For $A, B, C \in \Gamma(\mathfrak{sp}(V,\omega))$, the Jacobiator of $[\![-,-]\!]$ is the Jacobiator of the Lie algebroid $\Gamma(\mathfrak{sp}(V,\omega)$, which vanishes.
- For $A, B \in \Gamma(\mathfrak{sp}(V,\omega))$, $\omega_k \in \Gamma(\text{coker}(\phi_{k+1}^\omega))$, the Jacobiator being zero is equivalent to $\text{coker}(\phi_{k+1}^\omega)$ being an $\mathfrak{sp}(V,\omega)$-representation.

- For $A \in \Gamma(\mathfrak{sp}(V,\omega))$, $\omega_k \in \Gamma(\mathrm{coker}(\phi^\omega_{k+1}), \omega_l \in \Gamma(\mathrm{coker}(\phi^\omega_{l+1})$, the Jacobiator being zero is equivalent to the $\mathfrak{sp}(V,\omega)$-action on $\Gamma(\mathrm{coker}(\phi^\omega_j))$ being a derivation of $[\![-,-]\!]$ restricted to negative degrees. This is the case because r_k intertwines the $\mathfrak{sp}(V,\omega)$-actions on $\Gamma(\wedge^k V^* \otimes \wedge^2 V^*)$ and $\Gamma(\wedge^{k+1} V^* \otimes \wedge^2 V^*)$.

Consequently, the Jacobiator vanishes when at least one entry has degree 0. Now let $k,l,m \geq 2$, and $\omega_k \in \Gamma(\mathrm{coker}(\phi^\omega_{k+1}))$, $\omega_l \in \Gamma(\mathrm{coker}(\phi^\omega_{l+1})), \omega_m \in \Gamma(\mathrm{coker}(\phi^\omega_{m+1}))$. A lengthy computation shows that the Jacobiator

$$[\![[\![\omega_k,\omega_l]\!],\omega_m]\!] + (-1)^{(k-1)(l+m)}[\![[\![\omega_l,\omega_m]\!],\omega_k]\!] + (-1)^{(m-1)(k+l)}[\![[\![\omega_m,\omega_k]\!],\omega_l]\!]$$

does not vanish, but is equal to

$$\begin{aligned}\overline{\partial}([\![-,-,-]\!])(\omega_k,\omega_l,\omega_m) =& \overline{\partial}[\![\omega_k,\omega_l,\omega_m]\!] + [\![\overline{\partial}(\omega_k),\omega_l,\omega_m]\!] \\ &+ (-1)^{k-1}[\![\omega_k,\overline{\partial}(\omega_l),\omega_m]\!] + (-1)^{k+l}[\![\omega_k,\omega_l,\overline{\partial}(\omega_m)]\!],\end{aligned} \tag{2.17}$$

where $[\![\omega_k,\omega_l,\omega_m]\!]$ is given by the class of

$$\begin{aligned}&[r_{k+l-1}[\![\widehat{\omega_k,\omega_l}]\!],P_m(\omega_m)] + (-1)^{(k-1)(l+m)}[r_{k+l-1}[\![\widehat{\omega_l,\omega_m}]\!],P_k(\omega_k)] \\ &+ (-1)^{(m-1)(k+l)}[r_{k+l-1}[\![\widehat{\omega_m,\omega_k}]\!],P_l(\omega_l)]\end{aligned} \tag{2.18}$$

modulo the image of $\phi^\omega_{k+l+m-1}$. and

$$[\![\widehat{\omega_k,\omega_l}]\!] = [r_{k-1}\partial_k P_k(\omega_k), P_l(\omega_l)] + [P_k(\omega_k), r_{l-1}\partial_l P_l(\omega_l)] \in \Gamma(\wedge^{k+l-1}V^* \otimes \wedge^2 V^*).$$

We recognize equation (2.17) as a contracting homotopy for the Jacobiator: consequently, $-[\![-,-,-]\!]$ is a ternary operation satisfying the higher Jacobi identity an L_∞-algebroid must satisfy.

In particular, this does not equip the complex (2.14) with the structure of a dg-Lie algebroid as in the case of $\mathfrak{gl}(V)$, $\mathfrak{gl}(V,W)$, and $\mathfrak{sl}(V)$. As this structure is only unique up to L_∞-algebroid homotopy, this does of course not exclude the possibility that there exists a dg-Lie algebroid structure inducing the foliation $\mathcal{F}_\omega$.

In appendix 2.5.1, we investigate to what extent r^ω can be chosen to (anti)-commute with the differentials, as this would simplify both the binary and ternary bracket.

Remark 2.2.16.

- For degree reasons, the operation $[\![-,-,-]\!]$ vanishes when $\dim V \leq 4$. This means that when $\dim V = 2$ or $\dim V = 4$, the foliation $\mathcal{F}_\omega$ does admit a universal L_∞-algebroid with only a unary and binary bracket. Of course, for $\dim V = 2$, $\mathfrak{sp}(V,\omega) = \mathfrak{sl}(V)$, for which it was already known that a dg-Lie algebroid structure structure exists.

- When $\dim V = 6$, the unary, binary and ternary bracket determine the full L_∞-algebroid structure.

2.3 Higher-dimensional leaves

In this section we address question 2) of Section 2.1. Instead of considering foliations on a vector space with linear generators, we consider foliations $\mathcal{F}$ on vector *bundles* $\pi : E \to L$ which are generated by *fiberwise* linear vector fields such that the zero section is a leaf.

For $x \in L$ the fibers $E_x = \pi^{-1}(\{x\})$ of $\pi : E \to L$ are transverse to L, the foliation $\mathcal{F}$ restricts to the fibers ([AS09, Proposition 1.10]). We consider foliations for which the restriction $\mathcal{F}|_{E_x}$ coincides with one of the examples in Section 2.2. All results given in Section 2.2 will carry over, although in order to define the analogue of $\mathcal{F}_\mu$ and $\mathcal{F}_\omega$ we need the existence of non-vanishing sections of $\wedge^{\mathrm{rk}(E)}E^*$ and $\wedge^2 E^*$ respectively.

In Section 2.3.1, we consider the foliation of all vector fields on the vector bundle E, for which the restriction to the zero section is tangent to the zero section. This foliation consists of *all* fiberwise linear vector fields and is the analogue of $\mathcal{F}_0$ in Section 2.2.1.

In Section 2.3.2, we consider the foliation of all fiberwise linear vector fields on E tangent to a subbundle D, which is the analogue of $\mathcal{F}_W$ in Section 2.2.2.

In Section 2.3.3, we assume that E is orientable, and consider the foliation of all fiberwise linear vector fields on E which preserve a non-vanishing (on L) section $\mu \in \Gamma(\wedge^{\mathrm{rk}(E)}E^*)$, which is the analogue of $\mathcal{F}_\mu$ in Section 2.2.3.

In Section 2.3.4, we assume that $E \to L$ is a *symplectic* vector bundle with non-degenerate $\omega \in \Gamma(\wedge^2 E^*)$, and consider the foliation of all fiberwise linear vector fields on E which preserve ω, which is the analogue of $\mathcal{F}_\omega$ in Section 2.2.4.

2.3.1 Vector fields tangent to the zero section

To generalize Section 2.2.1, we need a generalization of the Lie algebra $\mathfrak{gl}(V)$ for a vector space V to a vector bundle. Let L be a smooth manifold, and let $\pi : E \to L$ be a (real) vector bundle of rank n. One way to generalize $\mathfrak{gl}(V)$ would be to use the Lie algebra bundle $\mathrm{End}(E)$. This is however not the right thing to consider: although it acts infinitesimally on E, for $x \in L$ the leaves of the foliation are given by $0_x \in E_x$ and the connected components of $E_x \setminus 0_x$. We are however interested in the situation where the zero section is a leaf, and the transverse foliation at a point of the zero section is given by (2.1). This is the case when dealing with a linearizable foliation around an embedded codimension n leaf with isotropy Lie algebra bundle $\mathrm{End}(E)$ acting on the n-dimensional fiber of the normal bundle by all vector bundle maps $E \to E$.
First recall that there are two distinguished classes of smooth functions on a vector bundle E. The fiberwise constant maps, given by the image of $\pi^* : C^\infty(L) \to C^\infty(E)$, and the fiberwise linear ones, given by $\Gamma(E^*)$.
Now let

$$\mathfrak{X}_{lin}(E) = \{X \in \mathfrak{X}(E) \mid X(\pi^*(C^\infty(L))) \subseteq \pi^*(C^\infty(L)),\ X(\Gamma(E^*)) \subseteq \Gamma(E^*)\}$$

be the set of vector fields preserving the fiberwise constant functions and fiberwise linear functions. First off, we note that $\mathfrak{X}_{lin}(E)$ is isomorphic to the sections of a transitive Lie algebroid over L (see [KSM02, Theorem 1.4]).

Lemma 2.3.1.

i) There is a short exact sequence of $C^\infty(L)$-modules

$$0 \longrightarrow \Gamma(End(E)) \xrightarrow{\ a\ } \mathfrak{X}_{lin}(E) \xrightarrow{\ \rho\ } \mathfrak{X}(L) \longrightarrow 0.$$

Here ρ is the restriction of a vector field to the subalgebra $\pi^*(C^\infty(L))$ and

$$a(A)(f) = \frac{d}{dt}\bigg|_{t=0} f \circ exp(-tA),$$

which is a fiberwise extension of the identification of linear vector fields on a vector space with the endomorphisms on the vector space.

ii) $\mathfrak{X}_{lin}(E)$ is a finitely generated projective $C^\infty(L)$-module. So there exists a vector bundle $\mathfrak{gl}(E)$ such that $\mathfrak{X}_{lin}(E) = \Gamma(\mathfrak{gl}(E))$.

iii) $\mathfrak{X}_{lin}(E)$ is closed under the Lie bracket of vector fields.

iv) The triple $(\mathfrak{gl}(E), \rho, [-,-])$ is a Lie algebroid.

Proof. i) is a local computation, ii) follows from i), and iii) and iv) are immediate. □

We can now construct a singular foliation on E for which L is a leaf: let

$$\mathcal{F}_L(E) = \{X \in \mathfrak{X}(E) \mid X|_L \in \mathfrak{X}(L)\}.$$

Lemma 2.3.2. *$\mathcal{F}_L(E) = Im(C^\infty(E) \otimes_{C^\infty(L)} \mathfrak{X}_{lin}(E) \overset{m}{\to} \mathfrak{X}(E))$, where m is the natural multiplication map.*

Proof. It follows from a local computation. □

Remark 2.3.3. Note that $C^\infty(E) \otimes_{C^\infty(L)} \mathfrak{X}_{lin}(E) = \Gamma(\pi^*(\mathfrak{gl}(E)))$, which are the sections of the action Lie algebroid corresponding to the natural action of $\mathfrak{X}_{lin}(E)$ on E. The anchor of this action Lie algebroid $\pi^*(\mathfrak{gl}(E))$ is m, and the bracket is given by

$$[f \otimes X, g \otimes Y] = fg \otimes [X,Y] + fX(g) \otimes Y - gY(f) \otimes X$$

for $f, g \in C^\infty(E)$, $X, Y \in \mathfrak{X}_{lin}(E)$.

2.3.1.1 The projective resolution

Lemma 2.3.2 now gives a first step in the resolution of $\mathcal{F}_L(E)$:

$$\Gamma(\pi^*(\mathfrak{gl}(E))) \xrightarrow{m} \mathcal{F}_L(E) \longrightarrow 0.$$

Observe that m is not injective! However, the kernel can be explicitly described. As a first step, we show that the kernel only affects the direction transverse to the leaf.

Lemma 2.3.4.

$$\ker(m) \subseteq \Gamma(\pi^*(End(E))) \subseteq \Gamma(\pi^*(\mathfrak{gl}(E))).$$

The following argument is thanks to Marco Zambon.

Proof. There is a commutative diagram of $C^\infty(E)$-modules given by

$$\begin{array}{ccc} \Gamma(\pi^*(\mathfrak{gl}(E))) & \xrightarrow{m} & \mathfrak{X}(E) \\ & \searrow^{\mathrm{Id}\otimes\rho} & \downarrow d\pi \\ & & \Gamma(\pi^*(TL)) \end{array} .$$

The statement now follows from Lemma 2.3.1i). □

Viewing $End(E)$ as $E^* \otimes E$, we can write down a complex analogous to (2.4).

$$0 \longrightarrow \Gamma(\pi^*(\wedge^n E^* \otimes E)) \xrightarrow{d_n} \dots \xrightarrow{d_2} \Gamma(\pi^*(\mathfrak{gl}(E))) \longrightarrow \mathcal{F}_L(E) \longrightarrow 0, \tag{2.19}$$

where

$$d_k : \Gamma(\pi^*(\wedge^k E^* \otimes E)) \to \Gamma(\pi^*(\wedge^{k-1} E^* \otimes E))$$

is given by

$$d_k(\alpha \otimes e) = y^i \iota_{e_i}(\alpha) \otimes e, \tag{2.20}$$

for $\alpha \otimes e \in \Gamma(\pi^*(\wedge^k(E^* \otimes E)))$, $\{e_i\}_{i=1}^n$ a local frame of E, and $\{y^i\}_{i=1}^n$ the corresponding linear coordinates (note that this does not depend on the choice of frame and defines a global section $\epsilon \in \Gamma(\pi^* E)$). We then have:

Proposition 2.3.5. *The complex* (2.19) *is exact.*

Proof. Proving exactness is completely analogous to the case considered in Section 2.2.1: it suffices to pick an open cover of L over which E trivializes (so $\pi^*(E)$ trivializes over the preimages of this cover), and show exactness over this open cover. But for trivial bundles the result is equivalent to the exactness of (2.4). □

2.3.1.2 The L_∞-algebroid structure

We claim that we can again find a dg-Lie algebroid structure on the resolution (2.19). Note that for degrees $-1, \dots, -n+1$, the involved spaces are simply the fiberwise extensions of (2.4), so we take the fiberwise extension of (2.5). To incorporate $\mathfrak{X}_{lin}(E)$ inside into this, we recall the following:

Lemma 2.3.6. *The action of $\mathfrak{X}_{lin}(E)$ on $\Gamma(E^*)$ extends to $\Gamma(E)$, all tensor, wedge and symmetric products and their pullbacks to E.*

Proof. Recall that an action of $\mathfrak{X}_{lin}(E)$ on a vector bundle F is a flat $\mathfrak{gl}(E)$-connection on the vector bundle F. As the action on $\Gamma(E^*)$ is equivalent to a $\mathfrak{gl}(E)$-connection on E^*, one can dualize this connection, and extend it via the Leibniz rule to tensor powers.
Finally, to extend the action to the pullback, we recall that $\Gamma(\pi^*(E^*)) = C^\infty(E) \otimes_{C^\infty(L)} \Gamma(E^*)$, and that both factors have a natural action of $\mathfrak{X}_{lin}(E)$. For $g \otimes X \in C^\infty(E) \otimes_{C^\infty(L)} \mathfrak{X}_{lin}(E)$, $f \otimes \alpha \in \Gamma(\pi^*(E^*))$, the action is given by

$$(g \otimes X) \cdot (f \otimes \alpha) = gX(f) \otimes \alpha + gf \otimes X(\alpha).$$

Since duals and tensors commute with pullbacks, the result follows. □

Using these actions, we can describe a dg-Lie algebroid structure on the resolution (2.19):

Proposition 2.3.7. *The complex* (2.19) *carries a dg-Lie algebroid structure, where the binary bracket is given by the analogue of equation* (2.5) *on elements of degree* -1 *and lower, and the bracket involving an element* $f \otimes X \in \Gamma(\pi^*(\mathfrak{gl}(E)))$ *and an element* $g \otimes \alpha \otimes e \in \Gamma(\pi^*(\wedge^k E^* \otimes E))$ *for* $f, g \in C^\infty(E)$*,* $X \in \mathfrak{X}_{lin}(E)$*,* $\alpha \in \Gamma(\wedge^k E^*)$*,* $e \in \Gamma(E)$ *is given by*

$$[f \otimes X, g \otimes \alpha \otimes e] = fX(g) \otimes \alpha \otimes e + fg \otimes X \cdot (\alpha \otimes e).$$

The bracket involving two elements of $\Gamma(\pi^*(\mathfrak{gl}(E)))$ *is given by the action Lie algebroid bracket.*

2.3.2 Linear vector fields preserving a subbundle

Let $\pi : E \to L$ be a vector bundle and $D \subseteq E$ a vector subbundle. In this section we combine Sections 2.2.2 and 2.3.1 to give a projective resolution of the subfoliation $\mathcal{F}_D \subseteq \mathcal{F}_L$ given by

$$\mathcal{F}_D(E) = \{X \in \mathcal{F}_L(E) \mid X(I_D) \subseteq I_D\},$$

where I_D is the vanishing ideal of $D \subseteq E$. In other words, $\mathcal{F}_D(E)$ consists of all vector fields which are tangent to the subbundle D. Note that when $D = 0$, we are in the situation of Section 2.3.1.
This can be approached in a similar way as $\mathcal{F}_L$: define

$$\mathfrak{X}_{lin}(E, D) := \{X \in \mathfrak{X}_{lin}(E) \mid X(\Gamma(\mathrm{Ann}(D))) \subseteq \Gamma(\mathrm{Ann}(D))\},$$

where $\Gamma(\mathrm{Ann}(D)) \subseteq \Gamma(E^*)$ is viewed as a subset of $C^\infty(E)$.

For $i \geq 2$, let $K_i \subseteq \wedge^i E^* \otimes E$ be the subbundle given by

$$K_i := \{\phi \in \wedge^i E^* \otimes E \mid \forall d \in D, \forall e_1, \ldots, e_{i-1} \in E : \phi(d, e_1, \ldots, e_{i-1}) \in D\},$$

Here the condition should be read fiberwise. Further, define $\mathfrak{gl}(E, D) \subseteq \mathfrak{gl}(E)$ as the subbundle whose sections are precisely $\mathfrak{X}_{lin}(E, D)$. Then the analogue of Proposition 2.2.4 holds, and we find:

Proposition 2.3.8.

i) For $j \geq 2$*, the differential*

$$d_j : \Gamma(\pi^*(\wedge^j E^* \otimes E)) \to \Gamma(\pi^*(\wedge^{j-1} E^* \otimes E))$$

as in (2.19) *restricts to a map*

$$d_j : \Gamma(\pi^*(K_j)) \to \Gamma(\pi^*(K_{j-1})),$$

and $d_2(\Gamma(\pi^*(K_2))) \subseteq \Gamma(\pi^*(\mathfrak{gl}(E,D)))$.

ii) The complex

$$\begin{array}{c} 0 \longrightarrow \Gamma(\pi^*(K_n)) \xrightarrow{d_n} \dots \\ \xrightarrow{d_2} \Gamma(\pi^*(\mathfrak{gl}(E,D))) \xrightarrow{\rho_D} \mathcal{F}_D(E) \longrightarrow 0 \end{array} \tag{2.21}$$

is exact.

iii) The bracket as described in Proposition 2.3.7 restricts to (2.21).

Consequently, (2.21) *with the restrictions of the differential and bracket is a universal L_∞-algebroid of the foliation $\mathcal{F}_D$, which is minimal at points of L.*

2.3.3 Vector fields preserving a volume form

In Section 2.2.3 we constructed the universal L_∞-algebroid for the foliation given by the action of $\mathfrak{sl}(V)$ for an n-dimensional vector space V. Although we made the choice of a volume form, this was not strictly necessary in this case.

Now if we want to generalize this example to the case of higher dimensional leaves, i.e. a linear foliation on a vector bundle $\pi : E \to L$ of rank n, such that the zero section is a leaf and the transverse foliation on the fibers E_x for $x \in L$ is isomorphic to the one given by the action of $\mathfrak{sl}(E_x)$, one approach would be to take an appropriate Lie subalgebroid $\mathfrak{sl}(E,\mu)$ of $\mathfrak{gl}(E)$ and to look at the induced foliation on E. To generalize the special linear subalgebra, the sections of the Lie subalgebroid $\mathfrak{sl}(E,\mu)$ should sit in a short exact sequence

$$0 \longrightarrow \Gamma(End_0(E)) \xrightarrow{a} \Gamma(\mathfrak{sl}(E,\mu)) \xrightarrow{\rho} \mathfrak{X}(L) \longrightarrow 0, \tag{2.22}$$

where $End_0(E)$ is the kernel of the vector bundle map $\mathrm{Tr} : End(E) \to \mathbb{R}$ given by the trace.

2.3.3.1 The projective resolution

Assume that E is orientable and pick a volume form $\mu \in \Gamma(\wedge^n E^*)$. Then we can identify the Lie algebroid $\mathfrak{sl}(E,\mu)$ as follows: Let

$$\mathfrak{X}^\mu_{lin}(E) = \{X \in \mathfrak{X}_{lin}(E) \mid X \cdot \mu = 0\},$$

where $X \cdot \mu$ is defined as in Lemma 2.3.6. A local computation shows that there exists a vector bundle $\mathfrak{sl}(E,\mu)$ satisfying (2.22) such that

$$\Gamma(\mathfrak{sl}(E,\mu)) = \mathfrak{X}^{\mu}_{lin}(E)$$

analogous to Lemma 2.3.1. To construct the projective resolution, we adopt a similar approach as in the case where L was a point. Define

$$\widehat{\mathrm{Tr}} : \mathfrak{X}_{lin}(E) \to \Gamma(\wedge^n E^*)$$

by

$$\widehat{\mathrm{Tr}}(X) = -X \cdot \mu.$$

Clearly, $\mathfrak{X}^{\mu}_{lin}(E) = \ker(\widehat{\mathrm{Tr}})$. Moreover, this really extends the trace:

Lemma 2.3.9. *For $A \in \Gamma(End(E))$,*

$$\widehat{\mathrm{Tr}} \circ a(A) = \mathrm{Tr}(A)\mu.$$

Proof. Pick a local frame $\{e_i\}_{i=1}^n$ for E and a dual frame $\{e^i\}_{i=1}^n$ for E^*, such that $\mu = e^1 \wedge \cdots \wedge e^n$. Then

$$\begin{aligned}
\widehat{\mathrm{Tr}}(a(A)) &= -\sum_{i=1}^n e^1 \wedge \cdots \wedge \left.\frac{d}{dt}\right|_{t=0} e^i \circ exp(-tA) \wedge \cdots \wedge e^n \\
&= -\sum_{i=1}^n e^1 \wedge \cdots \wedge e^i \circ \left.\frac{d}{dt}\right|_{t=0} exp(-tA) \wedge \cdots \wedge e^n \\
&= \sum_{i=1}^n e^1 \wedge \cdots \wedge e^i \circ A \wedge \cdots \wedge e^n \\
&= \mathrm{Tr}(A)\mu.
\end{aligned}$$

□

We now apply the same ideas as in the case where L is a point. As in Lemma 2.3.4, the kernel of $\rho_\mu : \Gamma(\pi^*(\mathfrak{sl}(E,\mu))) \to \mathfrak{X}(E)$ is contained in $\Gamma(\pi^*(\mathrm{End}_0(E)))$, so we proceed in a similar way as in Section 2.2.3. Define for $i = 2, \ldots, n$ the vector bundle map

$$\phi_i : \wedge^i E^* \otimes E \to \wedge^{i-1} E^*$$

over L by

$$\phi_i(\alpha \otimes e) = (-1)^{i-1} \iota_e(\alpha)$$

for $\alpha \in \wedge^i E^*, e \in E$. Setting $K_i = \ker(\phi_i)$, we obtain the following analog of Proposition 2.2.6:

Proposition 2.3.10.

$$0 \longrightarrow \Gamma(\pi^*(\wedge^n E^*)) \xrightarrow{d_n \phi_n^{-1} \partial_n} \Gamma(\pi^*(K_{n-1})) \xrightarrow{d_{n-1}} \dots \xrightarrow{d_2} \Gamma(\pi^*(\mathfrak{sl}(E,\mu))) \xrightarrow{\rho_\mu} \mathcal{F}_L^\mu(E) \longrightarrow 0 \tag{2.23}$$

is exact.

2.3.3.2 The L_∞-algebroid structure

As in Section 2.2.3.2, to define the L_∞-algebroid structure on the resolution (2.23), we would like to restrict the bracket as described in Proposition 2.3.7 to the kernel of the morphisms ϕ_k for $k \in \{1, \dots, n-1\}$. For elements of degrees -1 and lower and for two elements from $\Gamma(\pi^*(\mathfrak{sl}(E,\mu)))$ this is clear, as this is just the fiberwise extension of Lemma 2.2.7.
For the bracket between $\Gamma(\pi^*(\mathfrak{sl}(E,\mu)))$ and $\Gamma(\pi^*(K_q))$, take an element $f \otimes \alpha \otimes e \in \Gamma(K_q)$, where $f \in C^\infty(E), \alpha \in \Gamma(\wedge^q E^*), e \in \Gamma(E)$, and an element $X \in \Gamma(\pi^*(\mathfrak{sl}(E,\mu)))$, we compute

$$\begin{aligned}(-1)^{q-1}\phi_q([X, f \otimes \alpha \otimes e]) &= \rho_\mu(X)(f) \otimes \iota_e(\alpha) + f \otimes \iota_e(X \cdot \alpha) + f \otimes \iota_{X\cdot e}(\alpha) \\ &= \rho_\mu(X)(f)\iota_e(\alpha) + f \otimes X \cdot (\iota_e(\alpha)) \\ &= 0,\end{aligned}$$

as $f \otimes \alpha \otimes e \in \Gamma(K_q)$ means that $\iota_e(\alpha) = 0$. Hence, the bracket restricts to the subspaces given by the kernels of the ϕ_k. Finally, as in Section 2.2.3.2, we use the natural action of $\Gamma(\pi^*(\mathfrak{sl}(E,\mu)))$ on $\Gamma(\pi^*(\wedge^n E^*)) \cong C^\infty(E)\mu$ to define the bracket between degree 0 and $-n+1$.
Therefore, we again obtain a dg-Lie algebroid structure:

Proposition 2.3.11. *The resolution* (2.23) *of* $\mathcal{F}_L^\mu(E)$ *carries a dg-Lie algebroid structure, where the binary bracket is the restriction of the one described in Proposition 2.3.7 when both entries have degrees* $0, \dots, -n+2$*, and the bracket*

$$[X, \tau]$$

of $X \in \Gamma(\pi^*\mathfrak{sl}(E,\mu))$ *with* $\tau \in \Gamma(\pi^*(\wedge^n E^*))$ *is given by the natural action of* X *on* τ *as in Lemma 2.3.6.*

2.3.4 Linear vector fields preserving a fiberwise symplectic form

We now turn to the symplectic case: let $\pi : E \to L$ be a symplectic vector bundle with $\omega \in \Gamma(\wedge^2 E^*)$ a non-degenerate skew-symmetric bilinear form. By now we know how to construct a Lie subalgebroid of $\mathfrak{gl}(E)$ of linear vector fields preserving ω: consider

$$\mathfrak{X}^\omega_{lin}(E) := \{X \in \mathfrak{X}_{lin}(E) \mid X \cdot \omega = 0\},$$

where $X \cdot \omega$ is defined as in Lemma 2.3.6. Note that $\mathfrak{X}^\omega_{lin}(E)$ is closed under the Lie bracket of $\mathfrak{X}_{lin}(E)$.
As in the previous section, there exists a vector bundle $\mathfrak{sp}(E, \omega)$ over L, such that

$$\Gamma(\mathfrak{sp}(E, \omega)) = \mathfrak{X}^\omega_{lin}(E).$$

We therefore obtain a Lie subalgebroid $\mathfrak{sp}(E, \omega) \subseteq \mathfrak{gl}(E)$ over L, which generates a linear foliation $\mathcal{F}^\omega_L$ on E. The zero section is a leaf, and the transverse foliation on E_x for $x \in L$ is given by the standard $\mathfrak{sp}(E_x, \omega_x)$-action on E_x.

2.3.4.1 The projective resolution

We proceed as in Section 2.2.4. Define for $p = 1, \dots, n+1$ the vector bundle map

$$\phi^\omega_p : \wedge^p E^* \otimes E \to \wedge^{p-1} E^* \otimes \wedge^2 E^*$$

over L, given by

$$\phi^\omega_p(\alpha \otimes e) = (-1)^{p-1} \iota_{e_i}(\alpha) \otimes e^i \wedge \iota_e(\omega),$$

where $\{e_i\}_{i=1}^n$ and $\{e^i\}_{i=1}^n$ are dual local frames of E and E^* respectively. Note that ϕ^ω_p is independent of the choice of basis.

Define the differentials

$$d_p : \Gamma(\pi^*(\wedge^p E^* \otimes E)) \to \Gamma(\pi^*(\wedge^{p-1} E^* \otimes E))$$

as in equation (2.20), and

$$\partial_p : \Gamma(\pi^*(\wedge^p E^* \otimes \wedge^2 E^*)) \to \Gamma(\pi^*(\wedge^{p-1} E^* \otimes \wedge^2 E^*)),$$

given by

$$\partial_p(\alpha \otimes \tau) = -\iota_\epsilon(\alpha) \otimes \tau$$

for $\alpha \in \Gamma(\pi^*(\wedge^p E^*))$, $\tau \in \Gamma(\pi^*(\wedge^2 E^*))$. Then ϕ^ω is a cochain map of degree -1, and setting $C_i := \operatorname{coker}(\phi^\omega_i)$, with induced differentials $\overline{\partial_\bullet} : \Gamma(\pi^*(C_p)) \to \Gamma(\pi^*(C_{p-1}))$ we can describe a projective resolution as in Section 2.2.4:

Proposition 2.3.12. *The sequence*

$$0 \longrightarrow \Gamma(\pi^*(C_{n+1})) \xrightarrow{\overline{\partial_n}} \dots \xrightarrow{\overline{\partial_3}} \Gamma(\pi^*(C_3)) \longrightarrow \Gamma(\pi^*(\mathfrak{sp}(E,\omega))) \xrightarrow[\rho_\omega]{d_2\phi_2^{-1}\partial_2} \mathcal{F}_L^\omega(E) \longrightarrow 0 \tag{2.24}$$

is exact.

2.3.4.2 The (partial) L_∞-algebroid structure

To obtain the binary brackets, we follow the same approach as in Section 2.2.4. Let

$$r_p^\omega : \Gamma(\wedge^p E^* \otimes \wedge^2 E^*) \to \Gamma(\wedge^{p+1} E^* \otimes E)$$

be defined by

$$r_p^\omega(\alpha_p \otimes \tau) = \left(\frac{1}{p+1} \alpha_p \wedge \iota_{e_i}(\tau) - \frac{(-1)^p}{p(p+1)} \iota_{e_i}(\alpha_p) \wedge \tau \right) \otimes \omega^{-1}(e^i),$$

for $\alpha \in \Gamma(\wedge^p E^*), \tau \in \Gamma(\wedge^2 E^*)$. Then:

Proposition 2.3.13. *Using the notation from Section 2.2.4.2, the binary operation* $[\![-,-]\!]$ *on* (2.24) *defined by*

- *The standard* $\pi^*(\mathfrak{sp}(E,\omega))$*-action on itself and* $\pi^*(C_i)$ *for* $i = 3, \dots, n$ *when one of the entries lies in* $\Gamma(\pi^*(\mathfrak{sp}(E,\omega)))$,

- $$[\![\omega_p, \omega_q]\!] = [r_{p-1}^\omega \partial_p P_p(\omega_p), P_q(\omega_q)] + [P_p \omega_p, r_{q-1} \partial_q P_q(\omega_q)],$$

  for $\omega_p \in \Gamma(\pi^*(\wedge^p E^* \otimes \wedge^2 E^*)), \omega_q \in \Gamma(\pi^*(\wedge^q E^* \otimes \wedge^2 E^*))$ *where*

  $$P_p : \Gamma(\pi^*(\wedge^p E^* \otimes \wedge^2 E^*)) \to \Gamma(\pi^*(\ker(r_p^\omega)))$$

  is the projection $id - \phi_{p+1}^\omega \circ r_p^\omega$.

equips (2.24) *with a differential graded almost Lie algebroid structure, as in [LGLS20, Definition 3.68].*

The same remarks as in Section 2.2.4 can be made:

Remark 2.3.14.

- The operation defined in Proposition 2.3.13 satisfies the Jacobi identity if at least one entry has degree 0, but not when all of the entries have degree ≤ -1. The analogue of (2.18) defines a ternary operation which serves as a contracting homotopy for the Jacobiator.
- When the rank of E is at most 4, the Jacobi identity is trivially satisfied.
- When the rank of E is equal to 6, the full L_∞-algebroid structure is determined by the differential, the binary bracket as in Proposition 2.3.13 and the analogue of (2.18).

2.4 The isotropy L_∞-algebra in a fixed point

Let $\mathcal{F}$ be a foliation on the vector space V. Assume that the origin $p \in V$ be a leaf of $\mathcal{F}$. In [LGLS20, Section 4.2] the authors define an L_∞-algebra with trivial differential associated to a leaf of a foliation. Given a minimal resolution

$$0 \longrightarrow \Gamma(E_n) \xrightarrow{\partial_n} \dots \xrightarrow{\partial_1} \Gamma(E_0) \xrightarrow{\rho} \mathcal{F} \longrightarrow 0 \tag{2.25}$$

of $\mathcal{F}$ at p, and an L_∞-algebroid structure $\{\ell_k, \rho\}_{k\in\mathbb{N}}$ on $\Gamma(E_\bullet)$, it is defined by restricting the multibrackets ℓ_k to the fibers $(E_i)_p$, which is well-defined because $\rho_p = 0$. This L_∞-algebra is an invariant of $\mathcal{F}$, extending the isotropy Lie algebra $\mathcal{F}/I_p\mathcal{F}$, which is canonically isomorphic to $(E_0)_p$ with the restriction of ℓ_2 ([LGLS20, Proposition 4.14]). In particular, the binary bracket ℓ_2 turns $(E_i)_p$ into a $(E_0)_p$-representation. In this section we show that the spaces $(E_i)_p$ can be recovered directly from $\mathcal{F}$, without needing to find a projective resolution of $\mathcal{F}$. Moreover, we show that if $\mathcal{F}$ is linear, the $(E_0)_p$-representations on the $(E_i)_p$ can be determined explicitly. Note that the $(E_0)_p$-representation on $(E_0)_p$ is just the adjoint representation. This construction builds on [LGLS20, Remark 4.9] which states the following:

Lemma 2.4.1.
$$(E_i)_p \cong Tor_i^{C^\infty(V)}(\mathcal{F}, \mathbb{R}),$$
where the $C^\infty(V)$-module structure on $\mathbb{R}$ is defined by evaluation in the origin.

Proof. One way to construct $\mathrm{Tor}_i^{C^\infty(V)}(\mathcal{F}, \mathbb{R})$ is to take a projective resolution

$$0 \longrightarrow \Gamma(E_n) \xrightarrow{\partial_n} \dots \xrightarrow{\partial_1} \Gamma(E_0) \xrightarrow{\rho} \mathcal{F} \longrightarrow 0$$

of $\mathcal{F}$, then take the tensor product with $\mathbb{R}$ over $C^\infty(V)$ to obtain

$$0 \longrightarrow \Gamma(E_n) \otimes_{C^\infty(V)} \mathbb{R} \xrightarrow{\partial_n \otimes \mathrm{id}} \dots \xrightarrow{\partial_1 \otimes \mathrm{id}} \Gamma(E_0) \otimes_{C^\infty(V)} \mathbb{R} \longrightarrow 0$$

and compute the cohomology. As $\Gamma(E_i) \otimes_{C^\infty(V)} \mathbb{R} \cong (E_i)_p$ and the differentials become trivial, the result follows. □

It is however a well-known fact (see [Wei95, Theorem 2.7.2] for instance) that instead of first taking a projective resolution of $\mathcal{F}$ and then taking the tensor product with $\mathbb{R}$, we can equivalently first take a projective resolution of $\mathbb{R}$, and then take the tensor product with $\mathcal{F}$ and compute cohomology. A major advantage here is that we know an explicit resolution of $\mathbb{R}$: it is given by the complex (2.3). We therefore obtain:

Proposition 2.4.2. *For $i = 0, \dots, n$, we have*

$$(E_i)_p \cong H^i(\Gamma(\wedge^\bullet V^*) \otimes_{C^\infty(V)} \mathcal{F}(V), d_\bullet \otimes id) \tag{2.26}$$

where $\wedge^{-1}(V^*)$ is understood to be 0, and $d = \iota_{x^i \partial_{x^i}}$.

Next, we consider the action σ_i of the isotropy Lie algebra $\mathcal{F}/I_p\mathcal{F} \cong (E_0)_p$ on $(E_i)_p$ by the binary bracket ℓ_2. It turns out that when $\mathcal{F}$ is a *linear* foliation, we can define a canonical action on the right hand side of (2.26) which under the isomorphism of Proposition 2.4.2 corresponds to σ_i.

Proposition 2.4.3. *Let $\mathcal{F}$ be a linear foliation on V. Then:*

i) The map

$$lin : \mathcal{F} \to \mathcal{F}$$

given by

$$X \mapsto X^{(1)} \in \mathcal{F}$$

descends to an injective Lie algebra homomorphism

$$\overline{lin} : \mathcal{F}/I_p\mathcal{F} \to \mathcal{F}.$$

Here $X^{(1)}$ denotes the linear part of the vector field X.

ii) Let $i = 0, \dots, n$. For $X \in \mathcal{F}(V)$, $\alpha \otimes Y \in \Gamma(\wedge^i V^*) \otimes_{C^\infty(V)} \mathcal{F}(V)$, the assignment

$$(X + I_p\mathcal{F}) \cdot (\alpha \otimes \mathcal{F}) := [X^{(1)}, \alpha \otimes Y]_{FN} = \mathcal{L}_{X^{(1)}}(\alpha) \otimes Y + \alpha \otimes [X^{(1)}, Y] \tag{2.27}$$

defines a representation of $\mathcal{F}/I_p\mathcal{F}$ on $\Gamma(\wedge^i V^*) \otimes_{C^\infty(V)} \mathcal{F}(V)$, compatible with the differential $d_\bullet$, where $[-,-]_{FN}$ is the Frölicher-Nijenhuis bracket. Consequently, there is a well-defined action on the cohomology groups.

iii) The $\mathcal{F}/I_p\mathcal{F}$-action on $H^i(\Gamma(\wedge^\bullet V^*) \otimes_{C^\infty(V)} \mathcal{F}(V), d_\bullet \otimes id)$ induced by the $\mathcal{F}/I_p\mathcal{F}$-action (2.27) is equivalent to the $\mathcal{F}/I_p\mathcal{F}$-action on $(E_i)_p$.

Proof.

i) The fact that $\overline{lin}$ is a well-defined Lie algebra homomorphism was shown in [AZ14, Section 4]. For the injectivity, we need to show that if a vector field $Y \in \mathcal{F}(V)$ vanishes quadratically, it can be written as a linear combination

$$Y = \sum_{i=1}^{r} f^i X_i,$$

where $X_i \in \mathcal{F}(V)$, and $f^i(p) = 0$ for $i = 1, \ldots, r$. As $\mathcal{F}$ is a linear foliation, we can take the X_i to be linear vector fields which are linearly independent over $\mathbb{R}$. Then

$$0 = Y^{(1)} = \sum_{i=1}^{r} f^i(0) X_i,$$

which implies that $f^i(0) = 0$.

ii) This follows directly from the fact that $\overline{lin}$ is a Lie algebra homomorphism, and that the Frölicher-Nijenhuis bracket satisfies the Jacobi identity. The compatibility with the differentials follows from the fact that

$$[\mathcal{L}_{X^{(1)}}, \iota_{x^i \partial_{x^i}}] = \iota_{[X^{(1)}, x^i \partial_{x^i}]} = 0,$$

as $X^{(1)}$ is linear.

iii) For this, we recall the isomorphism (2.26) as described in [Wei95]. Given the projective resolutions (2.3) and (2.25), we can take the tensor product to obtain a double complex

$$(\Gamma(\wedge^\bullet V^*) \otimes_{C^\infty(V)} \Gamma(E_\bullet), d_\bullet \otimes \mathrm{id}, \mathrm{id} \otimes \partial_\bullet).$$

From the double complex, we can construct the total complex

$$(\mathrm{Tot}(\Gamma(\wedge^\bullet V^*) \otimes_{C^\infty(V)} \Gamma(E_\bullet), d_\bullet \otimes \mathrm{id} + \mathrm{id} \otimes \partial_\bullet).$$

Then the maps

$$\mathrm{id} \otimes \rho : \mathrm{Tot}(\Gamma(\wedge^\bullet V^*) \otimes_{C^\infty(V)} \Gamma(E_\bullet)) \to \Gamma(\wedge^\bullet V^*) \otimes_{C^\infty(V)} \mathcal{F}(V)$$

and

$$\mathrm{ev}_p \otimes \mathrm{id} : \mathrm{Tot}(\Gamma(\wedge^\bullet V^*) \otimes_{C^\infty(V)} \Gamma(E_\bullet)) \to \mathbb{R} \otimes_{C^\infty(V)} \Gamma(E_\bullet)$$

induce isomorphisms in cohomology. As both maps are compatible with the $\mathcal{F}/I_p\mathcal{F}$-actions, the isomorphisms in cohomology respect the $\mathcal{F}/I_p\mathcal{F}$-action as well.

□

The proposition now allows us to compute invariants of the foliation $\mathcal{F}(V)$ without needing an explicit resolution of $\mathcal{F}$, as we do in the following example.

Example 2.4.4. Consider on $V = \mathbb{R}^2$ the foliation $\mathcal{F}_1(V) = \langle x\partial_x, y\partial_x\rangle_{C^\infty(V)}$. Then by Lemma 2.4.1,

$$(E_0)_p := \mathrm{Tor}_0^{C^\infty(V)}(\mathcal{F}(V), \mathbb{R}) = \mathcal{F}_1/I_p\mathcal{F}_1.$$

For $\mathrm{Tor}_1^{C^\infty(V)}(\mathcal{F}(V), \mathbb{R})$, a straightforward computation shows that the middle cohomology of

$$\Gamma(\wedge^2 V^*) \otimes_{C^\infty(V)} \mathcal{F}(V) \xrightarrow{d_2} \Gamma(V^*) \otimes_{C^\infty(V)} \mathcal{F}(V) \xrightarrow{d_1} \mathcal{F}(V) \tag{2.28}$$

is one-dimensional, generated by the class of $\gamma := dx \otimes y\partial_x - dy \otimes x\partial_x$. Observe that this element is not exact in (2.28): although it can be written as

$$dx \otimes y\partial_x - dy \otimes x\partial_x = d_2(dx \wedge dy \otimes \partial_x)$$

in (2.4), $\partial_x \notin \mathcal{F}(V)$. Moreover, any exact element in (2.28) must vanish at least quadratically in the origin, which is not the case for γ.

Finally, it is easy to see that d_2 is injective, so we now know that for any minimal resolution (2.25), the space $(E_0)_p$ is two-dimensional, the space $(E_1)_p$ is one-dimensional, and the spaces $(E_i)_p$ for $i \geq 2$ are trivial. The Lie algebra structure on $(E_0)_p$ is the non-abelian two-dimensional Lie algebra, while the action of $(E_0)_p$ on $(E_1)_p$ is trivial.

Example 2.4.5. We can modify the previous example to obtain a foliation which is not linear: consider $\mathcal{F}_2(V) = \langle (x + xy)\partial_x + y^2\partial_y, y\partial_x\rangle_{C^\infty(V)}$. It is not difficult to see that $\mathcal{F}_2(V)$ is a projective $C^\infty(V)$-module. Consequently, for any minimal resolution (2.25), $(E_0)_p$ is two-dimensional, and $(E_i)_p = 0$ for $i \geq 1$. Although it was already known that there exists no analytic diffeomorphism of V taking the generators of $\mathcal{F}_2(V)$ to the generators of $\mathcal{F}_1(V)$ of the previous example (see [GS68, Proposition 1.2]), the above argument shows that there does not even exist a smooth diffeomorphism of V taking the $C^\infty(V)$-*module* $\mathcal{F}_2(V)$ to $\mathcal{F}_1(V)$, showing that not even the germs of the foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ are equivalent, even though the modules generated by the first order approximations of the generators around $p \in V$ are equal. Of course, in this case the difference between $\mathcal{F}_1(V)$ and $\mathcal{F}_2(V)$ can be seen by considering the dimension of the regular leaves: for $\mathcal{F}_1$ they are 1-dimensional, while for $\mathcal{F}_2$ they are 2-dimensional.

2.5 Appendix

2.5.1 Compatibility of r^ω with the differentials

In this section we use the notation from Section 2.2.4, and investigate whether the left inverse r^ω of ϕ^ω can be chosen to be a cochain map in some degrees, which would simplify the brackets of the L_∞-algebroid structure.

As the choice of r^ω in (2.16) is not unique, we investigate whether the left inverse r^ω can be chosen to be compatible with the differentials, as this would force $[\![-,-]\!]$ to be equal to $\{-,-\}$.
However, it is clear that this is not possible in all degrees: first of all, as r_1^ω is not only a left inverse, but the unique inverse, as ϕ_2^ω is an isomorphism. Hence, there is no choice there. Then, the existence of $\widetilde{r_2^\omega} : \Gamma(\wedge^2 V^* \otimes \wedge^2 V^*) \to \Gamma(\wedge^3 V^* \otimes V)$ such that

$$r_1^\omega \partial_2 + d_3 \widetilde{r_2^\omega} = 0$$

implies that $d_2 r_1^\omega \partial_2 = 0$, which is not the case.
Nevertheless, we consider the other degrees, as compatibility with the differentials would simplify the binary and ternary brackets.
We start with the lowest degree: Let $n = \dim V$. In degrees $-n$ and $-n+1$, we get the following square

$$\begin{array}{ccc} 0 & \longrightarrow & \Gamma(\wedge^n V^* \otimes V) \\ \downarrow & & \downarrow \phi_n^\omega \\ \Gamma(\wedge^n V^* \otimes \wedge^2 V^*) & \xrightarrow{\partial_n} & \Gamma(\wedge^{n-1} V^* \otimes \wedge^2 V^*) \end{array} .$$

Given a left inverse $\widetilde{r_{n-1}^\omega} : \Gamma(\wedge^{n-1} V^* \otimes \wedge^2 V^*) \to \Gamma(\wedge^n V^* \otimes V)$ of ϕ_n^ω such that

$$\widetilde{r_{n-1}^\omega} \partial_n = 0,$$

we note that the constant extension of the value at the origin $\widetilde{r_{n-1}^\omega}(0)$ is also a left inverse of ϕ_n^ω which satisfies

$$\widetilde{r_{n-1}^\omega}(0) \partial_n = 0.$$

It therefore suffices to show that there exists no constant (in V) left inverse $\widetilde{r_{n-1}^\omega}$ of ϕ_n^ω such that

$$\widetilde{r_{n-1}^\omega} \partial_n = 0.$$

Let $\mu \in \wedge^n V^*, \tau \in \wedge^2 V^*$. Then we can view ∂_n as an injective $\mathbb{R}$-linear map

$$\partial_n : \wedge^n V^* \otimes \wedge^2 V^* \to V^* \otimes \wedge^{n-1} V^* \otimes \wedge^2 V^*,$$

as ∂_n has linear coefficient functions, and $\widetilde{r^\omega_{n-1}}$ extends by $\mathrm{id}_{V^*} \otimes \widetilde{r^\omega_{n-1}}$ to a map

$$\mathrm{id}_{V^*} \otimes \widetilde{r^\omega_{n-1}} : V^* \otimes \wedge^{n-1} V^* \otimes \wedge^2 V^* \to V^* \otimes \wedge^n V^* \otimes V$$

by $C^\infty(V)$-linearity.
Now

$$\mathrm{id}_{V^*} \otimes \widetilde{r^\omega_{n-1}} \partial_n (\mu \otimes \tau) = 0$$

implies that

$$e^i \otimes \iota_{e_i}(\mu) \otimes \tau \in \ker(\mathrm{id}_{V^*} \otimes \widetilde{r^\omega_{n-1}}).$$

However, as $\ker(\mathrm{id}_{V^*} \otimes \widetilde{r^\omega_{n-1}}) = V^* \otimes \ker(\widetilde{r^\omega_{n-1}})$, it follows that

$$\iota_{e_i}(\mu) \otimes \tau \in \ker(\widetilde{r^\omega_{n-1}})$$

for each $i = 1, \dots, n$.
These elements actually generate the entirety of $\wedge^{n-1} V^* \otimes \wedge^2 V^*$, forcing $\widetilde{r^\omega_{n-1}} = 0$, contradicting the assumption that $\widetilde{r^\omega_{n-1}} \phi^\omega_n = \mathrm{id}$.
Now fix $\dim V = 4$. The general case discussed above shows that there exists no left inverse $\widetilde{r^\omega_3}$ of ϕ^ω_4 such that

$$d_4 \widetilde{r^\omega_3} \partial_4 = 0.$$

We will show that there exists no $\mathfrak{sp}(V, \omega)$-*equivariant* left inverse $\widetilde{r^\omega_2}$ of ϕ^ω_3 satisfying

$$d_3 \widetilde{r^\omega_2} \partial_3 = 0.$$

The requirement that $\widetilde{r^\omega_2}$ is $\mathfrak{sp}(V, \omega)$-equivariant is natural, as ϕ^ω_3 is. We follow [FH91, Chapter 16] to determine the space of all $\mathfrak{sp}(V, \omega)$-equivariant maps $\wedge^2 V^* \otimes \wedge^2 V^* \to \wedge^3 V^* \otimes V$, and then restrict to those which are left inverses of ϕ^ω_3. For this, we decompose the respective spaces into irreducible $\mathfrak{sp}(V, \omega)$-representations:

Lemma 2.5.1.

$$R_1 := \wedge^3 V^* \otimes V \cong \mathbb{R} \oplus W \oplus S^2(V)$$

$$R_2 := \wedge^2 V^* \otimes \wedge^2 V^* \cong \mathbb{R}^{\oplus 2} \oplus W^{\oplus 2} \oplus S^2(V) \oplus C,$$

where $W = Ann(\mathbb{R}\omega) \subseteq \wedge^2 V$, and C is an irreducible representation not isomorphic to $\mathbb{R}$, W or $S^2(V)$.

Now we would like to apply a variation of Schur's lemma (see for instance [Hum73]) to compute the space of $\mathfrak{sp}(V, \omega)$-equivariant maps $\wedge^2 V^* \otimes \wedge^2 V^* \to \wedge^3 V^* \otimes V$. We first obtain:

Lemma 2.5.2.

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{sp}(V,\omega)}(R_2, R_1) &\cong \mathrm{End}_{\mathfrak{sp}(V,\omega)}(\mathbb{R})^{\oplus 2} \oplus \mathrm{End}_{\mathfrak{sp}(V,\omega)}(W)^{\oplus 2} \oplus \mathrm{End}_{\mathfrak{sp}(V,\omega)}(S^2(V))\\ &\cong \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R}.\end{aligned}$$

Proof. By Schur's lemma, the restriction of a map of representation to irreducible factors is either 0, or an isomorphism, which proves the first isomorphism in the statement. For the second isomorphism, we observe that when complexifying, the representations

$$\mathbb{C}, W \otimes_{\mathbb{R}} \mathbb{C}, S^2_{\mathbb{C}}(V \otimes_{\mathbb{R}} \mathbb{C})$$

are irreducible $\mathfrak{sp}(V \otimes_{\mathbb{R}} \mathbb{C}, \omega)$-representations, where ω is now extended to a $\mathbb{C}$-bilinear skew-symmetric map

$$\omega : V \otimes_{\mathbb{R}} \mathbb{C} \times V \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C}.$$

Moreover, it is easy to see that for any representation T, the natural map

$$\mathrm{End}_{\mathfrak{sp}(V,\omega)}(T) \otimes_{\mathbb{R}} \mathbb{C} \to \mathrm{End}_{\mathfrak{sp}(V \otimes_{\mathbb{R}} \mathbb{C},\omega)}(T \otimes_{\mathbb{R}} \mathbb{C})$$

is an isomorphism. As the endomorphism ring of a complex irreducible representation is $\mathbb{C}$ by Schur's lemma, it follows that the endomorphism ring of the real representations $\mathbb{R}, W, S^2(V)$ is $\mathbb{R}$, concluding the proof of the lemma. □

We explicitly construct the generators of $\mathrm{Hom}_{\mathfrak{sp}(V,\omega)}(R_2, R_1)$: pick a basis $\{e_i\}_{i=1}^4$ of V such that $\omega = e^1 \wedge e^3 + e^2 \wedge e^4$, and let

$$\pi_\omega = \frac{1}{2}(e_1 \otimes e_3 + e_2 \otimes e_4 - e_3 \otimes e_1 - e_4 \otimes e_2) \in V \otimes V.$$

Lemma 2.5.3. *Let $\tau \in \wedge^2 V^*$. Define*

$$\overline{\tau} := \tau - \frac{1}{2}(\tau(e_1, e_3) + \tau(e_2, e_4))\omega.$$

Let $\tau_1, \tau_2 \in \wedge^2 V^*$. Then $\mathrm{Hom}_{\mathfrak{sp}(V,\omega)}(R_2, R_1)$ *is generated by the maps*

$$p_1(\tau_1 \otimes \tau_2) = \frac{1}{4}(\tau_1(e_1,e_3) + \tau_1(e_2,e_4))(\tau_2(e_1,e_3) + \tau_2(e_2,e_4))\pi_\omega,$$

$$p_2(\tau_1 \otimes \tau_2) = (\overline{\tau_1} \wedge \overline{\tau_2})(e_1,e_3,e_2,e_4)\pi_\omega,$$

$$q_1(\tau_1 \otimes \tau_2) = ((\omega^\flat)^{-1} \wedge (\omega^\flat)^{-1})(\overline{\tau_1})\frac{1}{2}(\tau_2(e_1,e_3) + \tau_2(e_2,e_4)),$$

$$q_2(\tau_1 \otimes \tau_2) = \frac{1}{2}(\tau_1(e_1,e_3) + \tau_1(e_2,e_4))((\omega^\flat)^{-1} \wedge (\omega^\flat)^{-1})(\overline{\tau_2}),$$

$$s(\tau_1 \otimes \tau_2) = \overline{\tau_1}((\omega^\flat)^{-1}(e^k), e_j)\overline{\tau_2}(e_k, e_l)\omega^{-1}(e^j) \cdot \omega^{-1}(e^l),$$

where p_1, p_2 correspond to the trivial representation, q_1, q_2 to W, and s to $S^2(V)$. Here $\cdot$ denotes the symmetric product in $S^2(V)$, $\wedge^3 V^* \otimes V$ is identified with $V \otimes V$ via the volume form $\frac{1}{2}\omega \wedge \omega$, and $\wedge^2 V$ and $S^2(V)$ sit inside $V \otimes V$ as

$$v_1 \wedge v_2 \mapsto \frac{1}{2}(v_1 \otimes v_2 - v_2 \otimes v_1),$$

$$v_1 \cdot v_2 \mapsto \frac{1}{2}(v_1 \otimes v_2 + v_2 \otimes v_1).$$

The lemma above allows us to formulate a condition under which

$$\lambda_1 p_1 + \lambda_2 p_2 + \mu_1 p_1 + \mu_2 p_2 + \nu s \tag{2.29}$$

is a left inverse of ϕ_3^ω:

Lemma 2.5.4. (2.29) *is a left inverse of ϕ_3^ω if and only if*

$$\lambda_1 = 2 - 10\lambda_2,$$

$$\mu_1 = \mu_2 - 2,$$

$$\nu = -2.$$

It is now straightforward to show that there is no value of $\lambda_2, \mu_2 \in \mathbb{R}$ such that the corresponding map $\widetilde{r_2^\omega} = (2 - 10\lambda_2)p_1 + \lambda_2 p_2 + (\mu_2 - 2)q_1 + \mu_2 q_2 - 2s$ satisfies

$$d_3 \widetilde{r_2^\omega} \partial_3 = 0.$$

Consequently:

Proposition 2.5.5. *When* $\dim V = 4$*, there exist no*

$$\widetilde{r_2^\omega} : \Gamma(\wedge^2 V^* \otimes \wedge^2 V^*) \to \Gamma(\wedge^3 V^* \otimes V), \widetilde{r_3^\omega} : \Gamma(\wedge^3 V^* \otimes \wedge^2 V^*) \to \Gamma(\wedge^4 V^* \otimes V)$$

satisfying

$$\widetilde{r_2^\omega}\partial_3 + d_4\widetilde{r_3^\omega} = 0.$$

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Chapter 3

Stability of fixed points in Poisson geometry and higher Lie theory

This chapter contains the article [Sin22].

Abstract - We provide a uniform approach to obtain sufficient criteria for a (higher order) fixed point of a given bracket structure on a manifold to be stable under perturbations. Examples of bracket structures include Lie algebroids, Lie n-algebroids, singular foliations, Lie bialgebroids, Courant algebroids and Dirac structures in split Courant algebroids admitting a Dirac complement. We in particular recover stability results of Crainic-Fernandes for zero-dimensional leaves, as well as the stability results of higher order singularities of Dufour-Wade.

These stability problems can all be shown to be specific instances of the following problem: given a differential graded Lie algebra $\mathfrak{g}$, a differential graded Lie subalgebra $\mathfrak{h}$ of finite codimension in $\mathfrak{g}$ and a Maurer-Cartan element $Q \in \mathfrak{h}^1$, when are Maurer-Cartan elements near Q in $\mathfrak{g}$ gauge equivalent to elements of $\mathfrak{h}^1$?

We show that the vanishing of a finite-dimensional cohomology group associated to $\mathfrak{g}, \mathfrak{h}$ and Q implies a positive answer to the question above, and therefore implies stability of fixed points of the geometric structures described above.

3.1 Introduction

In differential geometry, there are various structures of infinitesimal nature on a manifold M which induce a partition of M into immersed submanifolds called leaves. Examples include Lie algebra actions, involutive distributions and Poisson structures. These are all examples of Lie algebroids. A Lie algebroid over a manifold M is a vector bundle $A \to M$, equipped with a bundle map

$$\rho : A \to TM$$

covering the identity on M, together with a Lie bracket on the space of sections of A such that for every section $x, y \in \Gamma(A)$ and $f \in C^\infty(M)$, we have the equality

$$[x, fy]_A = \rho(x)(f)y + f[x, y]_A.$$

This property together with the Jacobi identity for $[-,-]_A$ implies that $\rho : \Gamma(A) \to \mathfrak{X}(M)$ is a Lie algebra map, therefore the image of ρ defines a singular foliation on M. As the space of Lie algebroid structures on a vector bundle A carries a topology, one can ask when a leaf L of a given Lie algebroid structure $(\rho, [-,-]_A)$ is stable when perturbing the Lie algebroid structure. More precisely:

Given a leaf $L \subseteq M$, when is it the case that every Lie algebroid structure near $(\rho, [-,-]_A)$ has a leaf near L which is diffeomorphic to L?

For compact leaves L, this question was first answered in [CF10]. Here it was shown that if the first cohomology of the Lie algebroid restricted to L with values in a certain representation vanishes, then for every Lie algebroid structure near the original one, there exists a *family of leaves* diffeomorphic to L. Moreover, when $A = T^*M$, the cotangent bundle of M, the authors prove a stronger result for when the same conclusion holds when restricting the class of Lie algebroid structures to only those coming from Poisson structures, as well as a separate result which guarantees the existence of a family of leaves *symplectomorphic* to a given one.
The results of [CF10] only depend on the first order approximation of $(\rho, [-,-]_A)$ around the leaf L. Let $L = \{p\} \subseteq M$ be a fixed point, that is, a zero-dimensional leaf, and assume that ρ has a higher order of vanishing in p. Then the cohomological assumptions of [CF10] will not be satisfied, hence nothing can be concluded about the stability of p. In this case the result was extended to give a criterium for stability of higher order fixed points of Lie algebroids and Poisson structures in [DW06], which guarantees the existence of a family of *higher order* fixed points near p.

Main results

In this article we focus on the case of fixed points. In Section 3.2 we first recall the stability result of [CF10] and [DW06] for first order fixed points of Lie algebroids, and prove it directly in terms of Lie algebroid data, instead of passing through the identification with fiberwise linear multivector fields on the dual vector bundle. In Section 3.3.2 we show that the stability question for zero-dimensional leaves is equivalent to a question about the comparison of Maurer-Cartan elements of differential graded Lie algebras and Maurer-Cartan elements of a chosen differential graded Lie subalgebra as explained below. Using this reformulation as motivation, we state and prove the main theorem, which gives a sufficient condition for the inclusion of the differential graded Lie subalgebra to be locally surjective on equivalence classes of Maurer-Cartan elements (Theorem 3.3.20). The rest of the article is dedicated to applying it in various situations by making appropriate choices for the differential graded Lie algebras involved. In Section 3.4 we apply the main theorem to obtain:

- the general result of [DW06] for Lie algebroids (Theorem 3.4.11),
- a stability result for (higher order) fixed points of Lie n-algebroids (Theorems 3.4.23 and 3.4.30), with an application to singular foliations (Section 3.4.5).

In Section 3.5 we apply the main theorem to obtain stability results for fixed points within a subclass of structures, such as:

- (higher order) fixed points of Lie algebroid structures on a vector bundle A^*, which form a Lie bialgebroid pair with a *given* Lie algebroid structure on A (Theorem 3.5.6),
- (higher order) fixed points of Poisson structures compatible with a fixed Nijenhuis tensor (Theorem 3.5.21),
- (higher order) fixed points of Courant algebroids (Theorem 3.5.36),
- fixed points of Dirac structures in split Courant algebroids admitting a Dirac complement (Theorem 3.5.50).

The results are all of the same form: given a structure Q as above, and a (higher order) fixed point $p \in M$ of this structure, there will be a finite-dimensional cohomology group associated to this structure and the (higher order) fixed point. If this cohomology group vanishes, then for every neighborhood V of $p \in M$, there exists a C^s-neighborhood $\mathcal{U}$ of the structure Q such that every

$Q' \in \mathcal{U}$ has a fixed point of the same kind in V. The precise value of $s \in \mathbb{Z}_{\geq 0}$ will depend on the type of structure and the order of the fixed point.

We approach these questions using the main theorem as follows. Let $p \in M$.

- The relevant structures are identified with Maurer-Cartan elements of a certain differential graded Lie algebra $(\mathfrak{g}, \partial, [-,-])$ given by the sections of some vector bundle.
- In here, we identify a differential graded Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, for which the Maurer-Cartan elements are those Maurer-Cartan elements of $\mathfrak{g}$ for which some *given* $p \in M$ is a fixed point of desired type. Let $Q \in \mathfrak{h}^1$ denote such a structure.
- $\mathfrak{g}^0$ acts on $\mathfrak{g}^1$ by the gauge action, as can be found in [Man04] for instance. In our examples, this action is by vector bundle automorphisms Φ covering a diffeomorphism ϕ of M. In particular, $p \in M$ is a fixed point of a Maurer-Cartan element Q' gauge transformed by $X \in \mathfrak{g}^0$ if and only if $\phi(p) \in M$ is a fixed point of the untransformed Q'.

The question of stability of p can now roughly be formulated as follows:

Given a Maurer-Cartan element Q of $\mathfrak{h}$, when is it true that any Maurer-Cartan element of $\mathfrak{g}$ near Q is gauge equivalent to a Maurer-Cartan element of $\mathfrak{h}$?

Under some conditions on $\mathfrak{g}$, $\mathfrak{h}$ and the gauge action, the most restrictive one being that $\mathfrak{g}^i/\mathfrak{h}^i$ is finite-dimensional for $i = 0, 1, 2$, we show that a sufficient condition for a positive answer is that

$$H^1(\mathfrak{g}/\mathfrak{h}, \overline{\partial + [Q,-]}) = 0.$$

Here $\overline{\partial + [Q,-]}$ is the induced map on the quotients. In several examples, this recovers known cohomology groups, and in all examples the chain complexes will consist of finite-dimensional vector spaces.
Where possible, we also point out relations between various structures and the cohomology groups that arise from them.
In a future work, we plan to explore the extent to which these results can be generalised so that they may be applied to higher-dimensional leaves.

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3.2 A direct proof of stability of fixed points of Lie algebroids

In this section we reprove the stability result of (first order) fixed points of Lie algebroids using the approach of [DW06], writing it directly in Lie algebroid data instead of passing through the identification with linear Poisson structures on the dual vector bundle. This proof already contains the key arguments to prove the main theorem of this article. We first give the definition of a Lie algebroid.

Definition 3.2.1. Let M be a smooth manifold. A *Lie algebroid over M* is a triple $(A, \rho, [-,-])$, where

i) $A \to M$ is a vector bundle,

ii) $\rho : A \to TM$ is a vector bundle map,

iii) $[-,-] : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ is an $\mathbb{R}$-bilinear skew-symmetric map,

such that

a) for all $x, y \in \Gamma(A)$, $f \in C^\infty(M)$, we have
$$[x, fy] = \rho(x)(f)y + f[x, y].$$

b) for all $x, y, z \in \Gamma(A)$,
$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

Lemma 3.2.2. *Property a) and b) imply that*
$$\rho([x, y]) = [\rho(x), \rho(y)]$$
for all $x, y \in \Gamma(A)$.

Recall that a Lie algebroid gives a partition of M in connected immersed submanifolds called leaves, such that the tangent space to the leaf through a point p coincides with the image of ρ at p.

Definition 3.2.3. Let M be a smooth manifold and let $(A, \rho, [-,-])$ be a Lie algebroid. A point $p \in M$ is a *fixed point of* $(\rho, [-,-])$ if $\rho_p = 0$.

Note that fixed points are exactly zero-dimensional leaves.
Above any point $p \in M$, the Lie algebra structure of $\Gamma(A)$ induces the structure of a Lie algebra on $\ker(\rho_p)$ called the *isotropy Lie algebra*.

Lemma 3.2.4. *Let $(A, \rho, [-,-])$ be a Lie algebroid over M, and let $p \in M$. Let*
$x, y \in \ker(\rho_p : A_p \to T_pM)$, and let $\tilde{x}, \tilde{y} \in \Gamma(A)$ be extensions of x, y respectively. Then the element

$$\mu_p(x,y) := [\tilde{x}, \tilde{y}](p)$$

is well-defined and lies in $\ker(\rho_p)$. Moreover, the map

$$\mu_p : \ker(\rho_p) \times \ker(\rho_p) \to \ker(\rho_p)$$

satisfies the Jacobi identity, equipping $\ker(\rho_p)$ with a Lie algebra structure.

For the rest of this section, assume that $p \in M$ is a fixed point of the Lie algebroid $(A, \rho, [-,-])$. Denote by $\mathfrak{g}_p$ the isotropy Lie algebra at p. As a vector space it is just A_p, while the Lie bracket is given as in Lemma 3.2.4.
The isotropy Lie algebra has a natural representation $\tau : \mathfrak{g}_p \to \mathrm{End}(T_pM)$ on T_pM, called the Bott representation or the linear holonomy representation. For $x \in \mathfrak{g}_p, v \in T_pM$, it is defined by

$$\tau(x)(v) = [\rho(\tilde{x}), \tilde{v}](p) \in T_pM, \tag{3.1}$$

where $\tilde{x} \in \Gamma(A), \tilde{v} \in \mathfrak{X}(M)$ are extensions of x and v respectively.
The cohomology of the isotropy Lie algebra $\mathfrak{g}_p$ with values in the linear holonomy representation τ on T_pM plays a vital role in Theorem 3.2.8 and will return several times throughout the article, so we recall the definition of Lie algebra cohomology.

Definition 3.2.5. Let $\mathfrak{g}$ be a Lie algebra, and let $\sigma : \mathfrak{g} \to \mathrm{End}(V)$ be a representation on a vector space V.

i) The *Chevalley-Eilenberg complex* is the cochain complex

$$(S(\mathfrak{g}^*[-1]) \otimes V, d^\sigma_{CE}),$$

where for $\alpha \in S^k(\mathfrak{g}^*[-1]) \otimes V$, $v_0, \dots, v_k \in \mathfrak{g}$, we have

$$d^\sigma_{CE}(\alpha)(v_0,\dots,v_k) = \sum_{i=0}^{k} (-1)^{i+k} \sigma(v_i)(\alpha(v_0, \dots, \widehat{v_i}, \dots, v_k)) + \sum_{0 \le i < j \le k} (-1)^{i+j+k} \alpha([v_i, v_j], v_0, \dots, \widehat{v_i}, \dots, \widehat{v_j}, \dots, v_k).$$

ii) When $V = \mathbb{R}$ and $\sigma = 0$, we denote by

$$(S(\mathfrak{g}^*[-1]), d_{CE})$$

the corresponding complex.

The following lemma is easy to show:

Lemma 3.2.6.

$$(S(\mathfrak{g}^*[-1]), d_{CE})$$

is a differential graded commutative algebra, and for any representation $\sigma : \mathfrak{g} \to \mathrm{End}(V)$*, the Chevalley-Eilenberg complex*

$$(S(\mathfrak{g}^*[-1]) \otimes V, d_{CE}^{\sigma})$$

defines a differential graded module over this algebra.

For our purposes, the relevant part of the complex is in degrees 0, 1 and 2. In these degrees, we denote the complex by

$$V \xrightarrow{d_0} \mathfrak{g}^*[-1] \otimes V \xrightarrow{d_1} S^2(\mathfrak{g}^*[-1]) \otimes V,$$

with

$$d_0(v)(x) = \sigma(x)(v), \tag{3.2}$$

and

$$d_1(\alpha)(x, y) = -\sigma(x)(\alpha(y)) + \sigma(y)(\alpha(x)) + \alpha([x, y]) \tag{3.3}$$

for $v \in V, \alpha \in \mathfrak{g}^*[-1] \otimes V, x, y \in \mathfrak{g}[1]$.
Denote the corresponding cohomologies by $H^i_{CE}(\mathfrak{g}, V)$ for $i = 0, 1$.

Remark 3.2.7.

i) This definition differs from the one which is mostly used by an overall factor of $(-1)^k$ when $\alpha \in S^k(\mathfrak{g}^*[-1]) \otimes V$. This is because we choose to write the complex using the shifted symmetric product rather than the unshifted wedge product.

ii) Note that for $H^i_{CE}(\mathfrak{g}, V)$ to be defined for $i = 0, 1$, it is not necessary that the bracket of $\mathfrak{g}$ satisfies the Jacobi identity. It is only needed that the bracket $[-, -] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ exists, and that

$$\sigma([x, y]) = \sigma(x)\sigma(y) - \sigma(y)\sigma(x)$$

for $x, y \in \mathfrak{g}$.

The last thing we need to formulate the stability result for fixed points of Lie algebroids is a topology on the space of Lie algebroid structures. It will be shown in Section 3.4.1 that Lie algebroid structures on A are elements of the sections of some vector bundle E, hence can be equipped with the weak C^k-topology induced by the one on $\Gamma(E)$. An element $Q \in \Gamma(E)$ can be seen as a pair $(\rho, [-, -])$, where ρ and $[-, -]$ are as in Definition 3.2.1, without requirement b). This is the description in terms of multiderivations as in [CM08].

Theorem 3.2.8 ([CF10],[DW06]). *Let M be a manifold and $(A, \rho, [-,-])$ be a Lie algebroid. Let $p \in M$ be a fixed point of $(A, \rho, [-,-])$. Let $\mathfrak{g}_p$ be the isotropy Lie algebra at p. Assume that*

$$H^1_{CE}(\mathfrak{g}_p, T_pM) = 0.$$

Then for any open neighborhood $U \subseteq M$ of p, there is a C^1-neighborhood $\mathcal{U}$ of $(\rho, [-,-])$ in the space of Lie algebroid structures such that for any $(\rho', [-,-]') \in \mathcal{U}$ there exists a family $I \subseteq U$ of fixed points of $(\rho', [-,-]')$ parametrized by an open neighborhood of the origin of $H^0_{CE}(\mathfrak{g}_p, T_pM)$.

Proof. The setup of the proof will be similar to the proof of Theorem 1.2 of [DW06]. The only difference is that we work with the description of Lie algebroids in terms of an anchor and a bracket, while the authors of [DW06] work with a special class of Poisson structures on A^*.
As the statement is local in M, we may assume that $M = V$ a finite-dimensional real vector space, $p = 0 \in V$, and that $A = \mathfrak{g}_p \times V$ is a trivial bundle.
The outline of the proof is as follows.

i) We construct a smooth map $\mathrm{ev}_Q : V \to W = \mathfrak{g}^*[-1] \otimes V$, parametrized continuously by elements of $Q \in \Gamma(E)$ with the C^1-topology, which contains the Lie algebroid structures on A.

ii) We construct a smooth map $R_{q,Q} : \mathfrak{g}^*[-1] \otimes V = W \to T = S^2(\mathfrak{g}^*[-1]) \otimes V$, parametrized continuously by $(q, Q) \in V \times \Gamma(E)$, where the second factor has the C^1-topology.

These maps will have the following properties: denoting by $Q_0 := (\rho, [-,-])$,

a) $\mathrm{ev}_{Q_0}(p) = 0$, and $(D(\mathrm{ev}_{Q_0}))_p = -d_0 : V \to W$, where d_0 is defined in equation (3.2) for the Bott representation τ. Moreover, if Q is a Lie algebroid structure, $\mathrm{ev}_Q(q) = 0$ if and only if q is a fixed point of Q.

b) $R_{q,Q}(0) = 0$ for every $(q, Q) \in V \times \Gamma(E)$, and $(D(R_{p,Q_0}))_0 = d_1$, with d_1 as in equation (3.3) for the Bott representation τ.

c) Whenever $Q \in \Gamma(E)$ is a Lie algebroid structure and $q \in V$, we have

$$R_{q,Q}(\mathrm{ev}_Q(q)) = 0 \in T.$$

The existence of the maps with these properties is sufficient to prove the theorem. Before we construct the maps, we show how the theorem follows from these properties. Let C be a complement to $\ker(d_1) = \mathrm{im}(d_0)$ in $W = \mathfrak{g}^*[-1] \otimes V$.

First property b) implies that R_{p,Q_0} restricted to C is an immersion at $0 \in C$ as C has trivial intersection with $\ker(d_1 = (D(R_{p,Q_0}))_0)$. By Lemma 3.6.3, there exists an open neighborhood O of $0 \in C$, an open neighborhood S of $p \in V$ and a C^1-neighborhood $\mathcal{U}_2$ of $Q_0 \in \Gamma(E)$ such that $R_{q,Q}\big|_O$ is an injective immersion for $Q \in \mathcal{U}_2$ and $q \in S$.
Property a) implies that ev_{Q_0} intersects $O \subseteq C$ transversely in p, as $d_0 = (D(\mathrm{ev}_{Q_0}))_p$, and $\mathrm{im}(d_0) = \ker(d_1)$ by the cohomological assumption. Therefore by Lemma 3.6.1, there exists a C^1-neighborhood of $Q_0 \in \mathcal{U}_1 \subseteq \Gamma(E)$ such that for any $Q \in \mathcal{U}_1$, there exists a $q \in S$ such that $\mathrm{ev}_Q(q) \in O$.
By property c), for any Lie algebroid structure $Q \in \mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2$, and any $q \in V$ we have

$$R_{q,Q}(\mathrm{ev}_Q(q)) = 0.$$

By injectivity of $R_{q,Q}$ restricted to O, combined with the fact that $\mathrm{ev}_Q(q) \in O$, it follows that

$$\mathrm{ev}_Q(q) = 0,$$

or equivalently, q is a fixed point of Q. For the existence of the family of fixed points, apply the final statement of Lemma 3.6.1, again using that $R_{q,Q}$ is injective restricted to O.
Now we define the maps.

i) For $Q = (\sigma, [-.-]')$, let

$$\mathrm{ev}_Q : V \to W$$

be defined by

$$\mathrm{ev}_Q(v) = T_v^*(\sigma)_p \in W,$$

where $T_v : V \to V$ is the translation by $v \in V$. The continuous dependence on $Q \in \Gamma(E)$ holds by definition of the topology on $\Gamma(E)$.

ii) Now let $q \in V$, and $Q = (\sigma, [-,-]') \in \Gamma(E)$. Define

$$R_{q,Q} : W \to T$$

by

$$R_{q,Q}(\alpha)(x,y) = -T_q^*([\sigma(x), \widetilde{\alpha}(\widetilde{y})] - [\sigma(y), \widetilde{\alpha}(\widetilde{x})] - \alpha([\widetilde{x}, \widetilde{y}]'))_p,$$

where $\alpha \in W$, $x, y \in \mathfrak{g}_p$, and the $\tilde{\cdot}$ indicates the extension of $\cdot$ to a constant section. We postpone the proof of the continuous dependence on $(q, Q) \in V \times \Gamma(E)$ to Section 3.4.1, as it will be shown more generally.

a) It is then clear that $\mathrm{ev}_{Q_0}(p) = 0$, and that q is a fixed point of Q if and only if $\mathrm{ev}_Q(q) = 0$. For the statement about the derivative of ev_{Q_0} note that the translation map $T_v : V \to V$ is the time-1 flow of the *constant* vector field induced by $v \in V$, so for $v \in V \cong T_pV, x \in A_p$ viewing both as constant sections, we have

$$(D(\mathrm{ev}_{Q_0}))_p(v)(x) = \frac{d}{dt}\bigg|_{t=0} T^*_{tv}(\rho)_p(x) = [v, \rho(x)](p) = -d_0(v)(x).$$

Here in the second to last equality we note that $T^*_{-tv}(\rho)_p(x)$ is the pushforward of the vector field $\rho(x)$ by the diffeomorphism $T_{tv} : V \to V$, which is the time t flow of the constant vector field v.

b) The properties regarding its value and derivative are immediate.

c) We note that

$$R_{q,Q}(\mathrm{ev}_Q(q)) = -T^*_q([\sigma(x), \sigma(y)] - \sigma([x, y]'))_p = 0.$$

□

Remark 3.2.9.

- Observe that for the proof it was only necessary that $\sigma : \Gamma(A) \to \mathfrak{X}(M)$ preserves the brackets, and the full Jacobi identity for $[-, -]'$ was not needed. So the theorem actually yields a stability criterium for *almost* Lie algebroid structures on A (see [LGLS20], Definition 3.7). For almost Lie algebroids, the requirement that the Jacobiator vanishes identically is replaced by the requirement that it is $C^\infty(M)$-multilinear. In this case, the fiber of A over a singular point $p \in M$ carries a bracket (which does not necessarily satisfy the Jacobi identity), and the action on T_pM still makes sense, so the cohomology $H^1_{CE}(\mathfrak{g}, T_pM)$ is well-defined according to remark 3.2.7ii).

- If $0 \in W$ is a regular value for ev_{Q_0}, the map $R_{\cdot,\cdot}$ would not be needed. However, for dimensional reasons this can only happen if the Lie algebroid A has rank 1. When the rank of A is 1, 0 being a regular value of ev_{Q_0} is equivalent to the cohomological assumption.

- The map $R_{q,Q}$ is linear. It is therefore unnecessary to use Lemma 3.6.3. The linearity is a consequence of the fact that constant vector fields commute, and in Theorem 3.3.20 there will be a quadratic term. As the arguments are very similar we choose to give the general argument here, and refer back to this in the proof of the main theorem.

3.3 The main theorem

In this section, we state and prove a generalization of Theorem 3.2.8, which is an algebraic statement about differential graded Lie algebras (Theorem 3.3.20). We give some basic background on differential graded Lie algebras in Section 3.3.1. To motivate the generalization, we characterize Lie algebroid structures in terms of certain elements in a graded Lie algebra, and show how the problem of stability of singular points can be reformulated in terms of this graded Lie algebra (Section 3.3.2). In Section 3.3, we then state the assumptions and prove Theorem 3.3.20.

3.3.1 Differential graded Lie algebras

Definition 3.3.1. A *differential graded Lie algebra* is a triple $(\mathfrak{g}, \partial, [-,-])$, where

i) $\mathfrak{g}$ is a non-negatively graded real vector space,

ii) $\partial : \mathfrak{g} \to \mathfrak{g}$ is a linear map of degree 1,

iii) $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a degree 0 graded skew-symmetric bilinear map,

satisfying

a) $\partial \circ \partial = 0$,

b) $\partial([x,y]) = [\partial(x), y] + (-1)^{|x|}[x, \partial(y)]$ for all $x, y \in \mathfrak{g}$ where $|x|$ denotes the degree of x,

c) $[[x,y],z] = [x,[y,z]] - (-1)^{|x||y|}[y,[x,z]]$ for $x, y, z \in \mathfrak{g}$, where $|x|$ and $|y|$ denote the degree of x and y respectively.

Any element $Q \in \mathfrak{g}^1$ can be used to twist the differential ∂, by considering the map

$$\partial + [Q, -].$$

The resulting triple $(\mathfrak{g}, \partial + [Q,-], [-,-])$ still satisfies property b) and c). It will in general not satisfy a), as

$$(\partial + [Q,-]) \circ (\partial + [Q,-]) = \left[\partial(Q) + \frac{1}{2}[Q,Q], -\right].$$

This leads us to the following definition.

Definition 3.3.2. Let $(\mathfrak{g}, \partial, [-,-])$ be a differential graded Lie algebra, and $Q \in \mathfrak{g}^1$ an element of degree 1. Q is a *Maurer-Cartan* element if

$$\partial(Q) + \frac{1}{2}[Q, Q] = 0 \in \mathfrak{g}^2.$$

Denote the set of all Maurer-Cartan elements of $\mathfrak{g}$ by $MC(\mathfrak{g})$.

Convention 3.3.3. Although the examples we consider may have nonzero negative degrees, we will implicitly set the negative degrees equal to zero. Note that this is harmless: the bracket of two elements of non-negative degrees has non-negative degree, and the differential respects this. We may use the bracket on negative degrees to define certain subspaces however.

We start with some examples.

Example 3.3.4.

1) We can consider two extreme cases: if $\partial = 0$, a differential graded Lie algebra $(\mathfrak{g}, 0, [-,-])$ is just a graded Lie algebra. If $[-,-] = 0$, a differential graded Lie algebra $(\mathfrak{g}, \partial, 0)$ is just a cochain complex. In the former case, a Maurer-Cartan element is simply a degree 1 element $Q \in \mathfrak{g}^1$ such that $[Q, Q] = 0$. In the latter, Maurer-Cartan elements are precisely 1-cocycles.

2) Let M be a smooth manifold. Then $(\Gamma(S(TM[-1])[1], 0, [-,-]_{SN})$, where $[-,-]_{SN}$ is the Schouten-Nijenhuis bracket, is the graded Lie algebra of multivector fields on M. Its Maurer-Cartan elements are $\pi \in \Gamma(\wedge^2 TM)$ such that $[\pi, \pi]_{SN} = 0$. These are precisely the Poisson bivectors.

3) Let M be a smooth manifold, and A a vector bundle over M. The space of multiderivations has a graded Lie algebra structure as described in Proposition 1 in [CM08], and its Maurer-Cartan elements are precisely the Lie algebroid structures on A.
An equivalent description is also given in [CM08], in which the bracket is more intuitive, which can be generalized to graded vector bundles. Borrowing notation from graded geometry, of which the basics can be found in [CS11], any vector bundle $A \to M$ gives rise to a graded manifold $A[1]$. Its functions are given by the non-negatively graded commutative algebra

$$C^\infty(A[1]) := \Gamma(S(A^*[-1])).$$

Now there is a natural graded Lie algebra associated to $A[1]$: it is the graded Lie algebra of graded derivations of $C^\infty(A[1])$, which we denote by

$$\mathfrak{X}(A[1]) := \mathrm{Der}_{\mathbb{R}}(C^\infty(A[1])).$$

The graded commutator bracket

$$[X, Y] = X \circ Y - (-1)^{|X||Y|} Y \circ X$$

equips $\mathfrak{X}(A[1])$ with a graded Lie algebra structure, where $X, Y \in \mathfrak{X}(A[1])$ of degrees $|X|, |Y|$ respectively. It was first observed by A. Vaintrob [Vai97] that a degree 1 derivation Q on $C^\infty(A[1])$ satisfying $[Q, Q] = 2Q^2 = 0$ is equivalent to the data of a Lie algebroid.

We will encounter more examples differential graded Lie algebras in this text. Another construction on differential graded Lie algebras we will need is the gauge action of $\mathfrak{g}^0$ on $\mathfrak{g}^1$. For nilpotent graded Lie algebras, the solution can be written down as an infinite sum which terminates, see for instance the formula in [Man04] above remark V.33. As the differential graded Lie algebras we will encounter will come with a notion of differentiable paths, we take remark V.33 in [Man04] as a definition.

Definition 3.3.5. Let $Q \in \mathfrak{g}^1$, and $v \in \mathfrak{g}^0$. Consider the initial value problem in $\mathfrak{g}^1$ given by

$$\frac{d}{dt}\gamma_t = [v, \gamma_t] - \partial(v), \gamma_0 = Q. \tag{3.4}$$

Assume that there exists a unique solution for $t \in [0, 1]$. Then the *gauge action* of v on Q is defined to be γ_1, and will be denoted by Q^v.

This action satisfies a property similar to the exponential map for Lie groups.

Lemma 3.3.6. *For $t \in [0, 1]$, we have $\gamma_t = Q^{tv}$. That is, the time t solution of the initial value problem associated to v is equal to the time 1 solution of the initial value problem associated to tv.*

One of the main properties of the gauge action in [Man04] is that it should preserve the Maurer-Cartan elements of $\mathfrak{g}$. While for the nilpotent case this can be proven, we will assume this.

3.3.2 Lie algebroid stability in terms of a graded Lie algebra

In this section we will reformulate the problem of stability of fixed points of Lie algebroid structures on $A \to M$, as well as the solution provided by Theorem 3.2.8 completely in terms of operations on the graded Lie algebra $\mathfrak{X}(A[1])$, as motivation for Theorem 3.3.20. We do this in four steps:

i) Identify Lie algebroid structures on A with Maurer-Cartan elements of $\mathfrak{X}(A[1])$.

ii) Identify a graded Lie subalgebra $\mathfrak{X}_{p,1}(A[1]) \subseteq \mathfrak{X}(A[1])$ whose Maurer-Cartan elements are Lie algebroid structures for which a *given* $p \in M$ is a fixed point.

iii) Use the gauge action of $\mathfrak{X}^0(A[1])$ to identify a neighborhood of $\mathfrak{X}^0(A[1])/\mathfrak{X}^0_{p,1}(A[1])$ with a neighborhood of $p \in M$, so that $q \in M$ near p will be a fixed point of a Lie algebroid structure Q if and only if Q is gauge equivalent to an element in $\mathfrak{X}^1_{p,1}(A[1])$.

iv) Interpret the cohomology appearing in Theorem 3.2.8 in terms of $\mathfrak{X}(A[1])$, $\mathfrak{X}_{p,1}(A[1])$ and $Q \in \mathfrak{X}^1_{p,1}(A[1])$, which is the Lie algebroid structure we start with.

We carry out the steps described above.

i) We start by recalling the bijection between Lie algebroid structures on a vector bundle A in terms of an anchor and a bracket as in Definition 3.2.1 and Lie algebroid structures defined using degree 1 vector fields on the graded manifold $A[1]$, due to T. Voronov [Vor10].

Lemma 3.3.7. *Let M be a manifold and $(A, \rho, [-,-]_A)$ a Lie algebroid. Then the Lie algebroid differential*

$$Q(\alpha)(X_0, \ldots, X_k) = \sum_{i=0}^{k} (-1)^{i+k} \rho(X_i)(\alpha(X_0, \ldots, \widehat{X_i}, \ldots, X_k)) \qquad (3.5)$$
$$+ \sum_{0 \le i < j \le k} (-1)^{i+j+k} \alpha([X_i, X_j]_A, X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_k)$$

for $\alpha \in \Gamma(S^k(A^*[-1])), X_0 \in X_k \in \Gamma(A)$ defines a degree 1 derivation of the graded algebra $\Gamma(S(A^*[-1]))$ satisfying $[Q, Q] = 2Q^2 = 0$.
Conversely, let Q be a degree 1 derivation of $\Gamma(S(A^*[-1]))$ satisfying $[Q, Q] = 0$. Identifying $\iota : \Gamma(A) \to \mathfrak{X}^{-1}(A[1])$ using the contraction map, the structure maps

$$\rho(X)(f) = -[Q, \iota_X](f),$$
$$[X, Y]_A = \iota^{-1}([[\iota_Y, Q], \iota_X])$$

for $f \in C^\infty(M), X, Y \in \Gamma(A)$ equip A with a Lie algebroid structure.

From now on, a Lie algebroid $(A, \rho, [-,-]_A)$ will be denoted by (A, Q) where Q is defined by equation (3.5).

ii) We now identify a graded Lie subalgebra of $\mathfrak{X}(A[1])$, whose Maurer-Cartan elements are precisely those Lie algebroid structures for which a *given* $p \in M$ is a fixed point. For that we start with an observation:

Lemma 3.3.8. *Let (A, Q) be a Lie algebroid over M. Then $p \in M$ is a fixed point of Q if and only if*

$$Q(C^\infty(M)) \subseteq I_p\Gamma(A^*[-1]), \tag{3.6}$$

where I_p denotes the vanishing ideal of $p \in M$.

Proof. For any $X \in \Gamma(A)$ and $f \in C^\infty(M)$ we have the equality

$$\rho(X)(f) = -\iota_X(Q(f)).$$

If $Q(f) \in I_p\Gamma(A^*[-1])$, we get that $\rho(X)(f) \in I_p$. As X and f are arbitrary, this implies that $\rho_p = 0 \in A_p^* \otimes T_pM$, so $p \in M$ is a fixed point of Q.
Conversely, if $\rho_p = 0 \in A_p^* \otimes T_pM$, then for every $a \in A_p$, $Q(f)_p(a) = 0$. As $a \in A_p$ and f are arbitrary, $Q(f)_p = 0$ hence $Q(f) \in I_p\Gamma(A^*[-1])$. □

Note that condition (3.6) defines a linear subspace of $\mathfrak{X}^1(A[1])$. This condition can be extended to other degrees, which then defines a graded Lie subalgebra of $(\mathfrak{X}(A[1]), 0, [-,-])$:

Lemma 3.3.9. *Let for $p \in M$, $k = 0, \ldots, \mathrm{rk}(A)$*

$$\mathfrak{X}^k_{p,1}(A[1]) := \{Q \in \mathfrak{X}^k(A[1]) \mid Q(C^\infty(M)) \subseteq I_p\Gamma(S^k(A^*[-1]))\}.$$

Then $(\mathfrak{X}_{p,1}(A[1]), 0, [-,-])$ is a graded Lie subalgebra of $(\mathfrak{X}(A[1]), 0, [-,-])$.

Proof. We need to show that

$$[\mathfrak{X}^k_{p,1}(A[1]), \mathfrak{X}^l_{p,1}(A[1])] \subseteq \mathfrak{X}^{k+l}_{p,1}(A[1]),$$

which is a straightforward computation. □

Remark 3.3.10. The subscript 1 indicates that it is the subspace of vector fields which have prescribed vanishing behavior up to first order in p. We will also encounter higher order vanishing conditions later.

iii) The next question is now how to keep track of a Lie algebroid structure having a fixed point at $q \neq p \in M$. This is where the gauge action of $\mathfrak{X}(A[1])$ comes into play. We unpack the definition and construct the solution of the gauge action.
Recall that in the proof of Theorem 3.2.8, we used the translation map T_v, which was the time-1 flow of the constant vector field $v \in \mathfrak{X}(V) \cong C^\infty(V, V)$. We now describe this in a coordinate free way. First we describe $\mathfrak{X}^0(A[1])$. The following lemma is well-known, for a proof we refer to the appendix of [ZZ13].

Lemma 3.3.11.

$$\mathfrak{X}^0(A[1]) \cong CDO(A^*) \cong CDO(A).$$

Now the main point is that covariant differential operators are infinitesimal vector bundle automorphisms, and can be integrated to vector bundle automorphisms covering a diffeomorphism of the base M. For details and the notation, see appendix 3.6.2.
Let $D \in CDO(A)$, assume that $\sigma(D) \in \mathfrak{X}(M)$ is complete and let

$$(\tilde{\Phi}^D_{-t})^* : \Gamma(S(A^*[-1])) \to \Gamma(S(A^*[-1]))$$

be the automorphism associated to it. As $\mathfrak{X}(A[1])$ is the set of derivations of the algebra $C^\infty(A[1]) = \Gamma(S(A^*[-1]))$, any automorphism of $C^\infty(A[1])$ induces an automorphism of $\mathfrak{X}(A[1])$ by conjugation: for $X \in \mathfrak{X}(A[1])$, it is defined by

$$(\tilde{\Phi}^D_{-t})_*(X) = (\tilde{\Phi}^D_t)^* \circ X \circ (\tilde{\Phi}^D_{-t})^*.$$

The automorphism $(\tilde{\Phi}^D_{-t})_*$ interacts nicely with the subalgebra $\mathfrak{X}_{p,1}(A[1])$:

Lemma 3.3.12. *Let Q be a Lie algebroid structure, and $p \in M$. The automorphism $(\tilde{\Phi}^D_{-1})_*$ satisfies*

$$(\tilde{\Phi}^D_{-1})_*(Q) \in \mathfrak{X}^1_{p,1}(A[1]) \iff Q \in \mathfrak{X}^1_{\phi^X_1(p),1}(A[1]).$$

Proof. Let $f \in C^\infty(M)$. Then

$$(\tilde{\Phi}^D_{-1})_*(Q)(f) = (\tilde{\Phi}^D_1)^*(Q(f \circ \phi^X_{-1})). \tag{3.7}$$

Now for the implication $\Rightarrow$, we assume that the left hand side lies in $I_p\Gamma(A^*)$. It follows that $Q(f \circ \phi^X_{-1})$ lies in

$$(\tilde{\Phi}^D_{-1})^*(I_p\Gamma(A^*)) = I_{\phi^X_1(p)}\Gamma(A^*)$$

As $(\phi^X_{-1})^*$ is surjective, the implication follows.
Conversely, for the implication $\Leftarrow$, it follows immediately that the right hand side of (3.7) lies in $I_p\Gamma(A^*)$, proving the lemma. □

It is not a coincidence that the notation for the automorphism of $\mathfrak{X}(A[1])$ associated to an element $D \in CDO(A) \cong \mathfrak{X}^0(A[1])$ resembles the notation for the pushforward of a vector field along a diffeomorphism. The following lemma shows that this automorphism is precisely the gauge transformation by D^*:

Lemma 3.3.13. *Let $Q \in \mathfrak{X}(A[1])$, and $D \in CDO(A)$ with symbol X. Then whenever ϕ_t^X is defined, we have the equality*

$$\frac{d}{dt}(\tilde{\Phi}^D_{-t})_*(Q) = [D^*, (\tilde{\Phi}^D_{-t})_*(Q)]$$

with $(\tilde{\Phi}^D_0)_*(Q) = Q$.

We can now rephrase the question of stability in a way which only involves operations intrinsic to the graded Lie algebra $\mathfrak{X}(A[1])$. For simplicity, we write $\mathfrak{X}(A[1]) =: \mathfrak{g}$, $\mathfrak{X}_{p,1}(A[1]) =: \mathfrak{h}$, and for $Q \in \mathfrak{g}$, $D \in \mathfrak{g}^0$, we write

$$Q^D := (\tilde{\Phi}^D_{-1})_*(Q).$$

The stability problem can now roughly be formulated as:

Let (A, Q) be a Lie algebroid over M, and $p \in M$ a fixed point. When is it the case that for any Lie algebroid structure Q' near Q, there exists a $D^* \in \mathfrak{g}^0$ such that the solution $\gamma : [0,1] \to \mathfrak{X}^1(A[1])$ of the initial value problem

$$\frac{d}{dt}\gamma_t = [D^*, \gamma_t], \gamma_0 = Q'$$

satisfies $\gamma_1 \in \mathfrak{h}^1$?

Note that if $D \in \mathfrak{h}^0 \subseteq \mathfrak{g}^0$, then $Q^D \in \mathfrak{h}$ if and only if $Q \in \mathfrak{h}$. We may therefore restrict our search for such a $D^* \in \mathfrak{g}^0$ to a *complement* of $\mathfrak{h}^0$ in $\mathfrak{g}^0$. Such a complement is naturally isomorphic to T_pM. For our purposes however, it will be more convenient to work with the quotient $\mathfrak{g}^0/\mathfrak{h}^0$, with a chosen $\mathbb{R}$-linear splitting $\Sigma : \mathfrak{g}^0/\mathfrak{h}^0 \cong T_pM \to \mathfrak{g}^0$. As we are interested in small neighborhoods of the point p and will only look at the action by elements of the image of this splitting, the requirement that the symbol of a differential operator is a complete vector field is not restrictive. Indeed, on a coordinate chart where A trivializes, it is easy to see that the constant extension of an element of T_pM defines a splitting.

iv) Now that we have phrased the problem in this context, there is also a way to phrase the answer provided by Theorem 3.2.8 in this context. In fact, the cochain complex associated to the Bott representation arises naturally:

Lemma 3.3.14. *Let (A, Q) be a Lie algebroid over M, and $p \in M$ a fixed point, with isotropy Lie algebra $\mathfrak{g}_p$. Let $(\mathfrak{X}(A[1]), [Q, -], [-, -])$ and $(\mathfrak{X}_{p,1}(A[1]), [Q, -], [-, -])$ the associated differential graded Lie algebras. For $k = 0, \dots, \mathrm{rk}(A) = r$, there is a natural isomorphism*

$$\mathfrak{X}^k(A[1])/\mathfrak{X}^k_{p,1}(A[1]) \to S^k(\mathfrak{g}_p^*[-1]) \otimes T_pM$$

which intertwines the differential $\overline{[Q,-]}$ *induced on the quotient complex* $(\mathfrak{X}(A[1])/\mathfrak{X}_{p,1}(A[1]), \overline{[Q,-]})$ *with the Chevalley-Eilenberg differential* d^{τ}_{CE} *on the right hand side.*

Proof. There is a short exact sequence of graded $C^\infty(M)$-modules in which $\mathfrak{X}(A[1])$ sits:

$$0 \longrightarrow C^\infty(A[1]) \otimes \Gamma(A[1]) \xrightarrow{\iota} \mathfrak{X}(A[1]) \xrightarrow{\sigma} C^\infty(A[1]) \otimes \mathfrak{X}(M) \longrightarrow 0,$$

where ι denotes the contraction and σ the restriction to $C^\infty(M) \subseteq C^\infty(A[1])$.
Note that this shows that $\mathfrak{X}(A[1]) = \Gamma(E)$ for some graded vector bundle E, as any connection on A splits the sequence. Now the graded Lie algebra $\mathfrak{X}_{p,1}(A[1])$ also sits inside a short exact sequence of graded $C^\infty(M)$-modules:

$$0 \longrightarrow C^\infty(A[1]) \otimes \Gamma(A[1]) \xrightarrow{\iota} \mathfrak{X}_{p,1}(A[1]) \xrightarrow{\sigma} I_p C^\infty(A[1]) \otimes \mathfrak{X}(M) \longrightarrow 0$$

Consequently, on the quotients we get a short exact sequence of vector spaces

$$0 \longrightarrow 0 \longrightarrow \mathfrak{X}(A[1])/\mathfrak{X}_{p,1}(A[1]) \xrightarrow{\overline{\sigma}} S(\mathfrak{g}_p^*[-1]) \otimes T_pM \longrightarrow 0,$$

proving the isomorphism.
Note that by construction of the Lie algebra structure on $\mathfrak{g}_p$, we have for any $\alpha \in C^\infty(A[1])$ the equality

$$Q(\alpha)(p) = d_{CE}(\alpha(p)) \in \mathfrak{g}_p^*.$$

Now if $X \in \mathfrak{X}(A[1])$ and $\alpha \in C^\infty(A[1])$, then we know

$$\begin{aligned} d^{\tau}_{CE}(\alpha(p)\overline{\sigma}(X)) &= d_{CE}(\alpha(p))\overline{\sigma}(X) + (-1)^{|\alpha|}\alpha(p) d^{\tau}_{CE}(\overline{\sigma}(X)) \\ &= Q(\alpha)(p)\overline{\sigma}(X) + (-1)^{|\alpha|}\alpha(p) d^{\tau}_{CE}(\overline{\sigma}(X)) \\ &= \overline{\sigma}([Q,\alpha X]) + (-1)^{|\alpha|}\alpha(p)(d^{\tau}_{CE}(\overline{\sigma}(X)) - \overline{\sigma}([Q,X])). \end{aligned}$$

It therefore suffices to show the compatibility of differentials in degree 0. Let $X \in \mathfrak{X}^0(A[1])$, and $f \in C^\infty(M)$.
We interpret elements of $\alpha \in \mathfrak{g}_p^* \otimes T_pM$ as $\mathbb{R}$-bilinear maps $\alpha : \mathfrak{g}_p \times C^\infty(M) \to \mathbb{R}$, satisfying

$$\alpha(x, fg) = \alpha(x, f)g(p) + f(p)\alpha(x, g),$$

for $x \in \mathfrak{g}_p$, $f, g \in C^\infty(M)$. Then for $x \in \mathfrak{g}_p$, $f \in C^\infty(M)$,

$$\begin{aligned} \overline{\sigma}([Q, X])(x, f) &= Q(X(f))(p)(x) - X(Q(f))(p)(x) \\ &= \rho_p(x)(X(f)) - X(Q(f)(\tilde{x}))(p) \\ &= [\rho(\tilde{x}), \widetilde{\overline{\sigma}(X)}](f)(p) \\ &= d_{CE}(\overline{\sigma}(X))(x)(f). \end{aligned}$$

Here the $\tilde{}$ indicate some extension of $x, \overline{\sigma}(X)$ to sections of A and TM respectively, and the second and third equality hold because $p \in M$ is a fixed point of Q. □

For future reference, we state the short exact sequence as a lemma.

Lemma 3.3.15. *There is a short exact sequence of graded $C^\infty(M)$-modules*

$$0 \longrightarrow C^\infty(A[1]) \otimes \Gamma(A[1]) \xrightarrow{\iota} \mathfrak{X}(A[1]) \xrightarrow{\sigma} C^\infty(A[1]) \otimes \mathfrak{X}(M) \longrightarrow 0.$$

Moreover, any connection on A induces a splitting of the sequence.

This section can now be summarized to give an alternative formulation of Theorem 3.2.8, which involves only operations on the graded Lie algebra. Let $\mathfrak{g} = \mathfrak{X}(A[1]), \mathfrak{h} = \mathfrak{X}_{p,1}(A[1])$ as above, and let $\Sigma : \mathfrak{g}^0/\mathfrak{h}^0 \to \mathfrak{g}^0$ denote a splitting of the quotient map.

Theorem 3.3.16 (Reformulation of Theorem 3.2.8)**.** *Let $Q \in \mathfrak{h}$ be a Maurer-Cartan element of $\mathfrak{h}$ (hence of $\mathfrak{g}$). If*

$$H^1(\mathfrak{g}/\mathfrak{h}, \overline{[Q, -]}) = 0,$$

then for every open neighborhood V of $0 \in \mathfrak{g}^0/\mathfrak{h}^0 \cong T_pM$ there exists a C^1-open neighborhood $\mathcal{U}$ of Q in the space of Maurer-Cartan elements of $(\mathfrak{g}, 0, [-,-])$ such that for any $Q' \in \mathcal{U}$, there exists a family $I \subseteq V$, parametrized by an open neighborhood of the origin of $H^0(\mathfrak{g}/\mathfrak{h}, \overline{[Q, -]})$, with $(Q')^{\Sigma(v)} \in \mathfrak{h}^1$ for all $v \in I$.

Under some assumptions on $\mathfrak{g}$ and $\mathfrak{h}$, this theorem will hold in more generality. Making appropriate choices for $\mathfrak{g}$ and $\mathfrak{h}$ will then yield similar results in other contexts.

3.3.3 The main theorem: assumptions and proof

In this section we state some assumptions on differential graded Lie algebras $\mathfrak{h} \subseteq \mathfrak{g}$ so that Theorem 3.3.16 holds in more generality, and prove this general result.

Assumptions 3.3.17. Assume we have the following:

i) A differential graded Lie algebra $(\mathfrak{g}, \partial, [-,-])$,

ii) a differential graded Lie subalgebra $(\mathfrak{h}, \partial, [-,-])$ such that $\mathfrak{g}^i/\mathfrak{h}^i$ is finite-dimensional for $i = 0, 1, 2$,

iii) splittings $\sigma_i : \mathfrak{g}^i/\mathfrak{h}^i \to \mathfrak{g}^i$ for $i = 0, 1$,

iv) a Maurer-Cartan element $Q \in \mathfrak{h}^1 \subseteq \mathfrak{g}^1$,

such that

a) $\mathfrak{g}^i$ for $i = 0, 1, 2$ are locally convex topological vector spaces such that the projections $p_i : \mathfrak{g}^i \to \mathfrak{g}^i/\mathfrak{h}^i$ are continuous,

b) $\partial : \mathfrak{g}^1 \to \mathfrak{g}^2$ is continuous,

c) $[-,-] : \mathfrak{g}^1 \times \mathfrak{g}^1 \to \mathfrak{g}^2$ is continuous,

d) There is an open neighborhood U of $0 \in \mathfrak{g}^0/\mathfrak{h}^0$ such that for any $Q \in \mathfrak{g}^1$, the gauge action as in Definition 3.4 of $\sigma_0(v)$ for $v \in U$ on Q is defined, the assignment
$$U \times \mathfrak{g}^1 \ni (v, Q') \mapsto (Q')^{\sigma_0(v)} \in \mathfrak{g}^1$$
is jointly continuous, and its class mod $\mathfrak{h}^1$ depends smoothly on $v \in U$ for each fixed Q'.

e) For $v \in U$, $Q' \in \mathfrak{g}^1$ is Maurer-Cartan if and only if $(Q')^{\sigma_0(v)}$ is Maurer-Cartan.

Remark 3.3.18. The choice of Q implies that $(\mathfrak{g}, \partial + [Q,-])$ is a cochain complex, with $(\mathfrak{h}, \partial + [Q,-])$ as a subcomplex. We can therefore take the quotient complex which we denote by $(\mathfrak{g}/\mathfrak{h}, \overline{\partial + [Q,-]})$.

The following lemma gives a sufficient condition for condition e) of assumptions 3.3.17 to be satisfied. In particular, the lemma applies when $\mathfrak{g}$ is degreewise given by the sections of some vector bundle, and the bracket $[-,-]$ is a first order bidifferential operator.

Lemma 3.3.19. *Let $U \subseteq \mathfrak{g}^0/\mathfrak{h}^0$ as in d) of assumptions 3.3.17. If $v \in U$ and the initial value problem*

$$\frac{d}{dt}\gamma_t = [\sigma_0(v), \gamma_t], \gamma_0 = 0 \in \mathfrak{g}^2, \tag{3.8}$$

has only the constant solution $\gamma_t \equiv 0 \in \mathfrak{g}^2$, then $(Q')^{\sigma_0(v)} \in \mathfrak{g}^1$ is a Maurer-Cartan element if and only if $Q' \in \mathfrak{g}^1$ is.

Proof. Let $\alpha : [0, 1] \to \mathfrak{g}^1$ be a solution to equation (3.4), where $\alpha_0 = Q'$ is a Maurer-Cartan element. The expression

$$\gamma_t := \partial\alpha_t + \frac{1}{2}[\alpha_t, \alpha_t]$$

then satisfies the initial value problem (3.8), hence must be identically 0. □

We can now state the main theorem, which roughly states that given a Maurer-Cartan element $Q \in \mathfrak{h}^1$, if a certain cohomology group vanishes, every Maurer-Cartan element Q' of $\mathfrak{g}$ near Q is gauge equivalent to a Maurer-Cartan element of $\mathfrak{h}$. Moreover, it also describes the amount of different gauge equivalences that take Q' into $\mathfrak{h}$.

Theorem 3.3.20. *Assume that we are in the setting as described in assumptions 3.3.17. Assume that*

$$H^1(\mathfrak{g}/\mathfrak{h}, \overline{\partial + [Q, -]}) = 0.$$

Then for every open neighborhood V of $0 \in U$ there exists an open neighborhood $\mathcal{U} \subseteq MC(\mathfrak{g})$ of Q such that for any $Q' \in \mathcal{U}$ there exists a family I in V parametrized by an open neighborhood of $0 \in H^0(\mathfrak{g}/\mathfrak{h}, \overline{\partial + [Q, -]})$ with $(Q')^{\sigma(v)} \in \mathfrak{h}^1$ for $v \in I$.

Proof. The proof setup is similar to the proof of Theorem 3.2.8, the difference being in the maps which are used. We repeat the key steps of the proof.

i) Construct a smooth map $\mathrm{ev}_{Q'} : V \to \mathfrak{g}^1/\mathfrak{h}^1$ depending continuously on $Q' \in \mathfrak{g}^1$,

ii) construct a smooth map $R_{v,Q'} : \mathfrak{g}^1/\mathfrak{h}^1 \to \mathfrak{g}^2/\mathfrak{h}^2$ depending continuously on $(v, Q') \in V \times \mathfrak{g}^1$,

satisfying

a) $\mathrm{ev}_Q(0) = 0 \in \mathfrak{g}^1/\mathfrak{h}^1$, and $(D(\mathrm{ev}_Q))_0 = -(\overline{\partial + [Q,-]}) : \mathfrak{g}^0/\mathfrak{h}^0 \cong T_0 V \to T_0\mathfrak{g}^1/\mathfrak{h}^1 \cong \mathfrak{g}^1/\mathfrak{h}^1$. Moreover, $(Q')^{\sigma_0(v)} \in \mathfrak{h}^1$ if and only if $\mathrm{ev}_{Q'}(v) = 0 \in \mathfrak{g}^1/\mathfrak{h}^1$.

b) $R_{v,Q'}(0) = 0 \in \mathfrak{g}^2/\mathfrak{h}^2$ for every $(v, Q') \in V \times \mathfrak{g}^1$, and $(D(R_{0,Q}))_0 = \overline{\partial + [Q,-]} : \mathfrak{g}^1/\mathfrak{h}^1 \to \mathfrak{g}^2/\mathfrak{h}^2$.

c) Whenever $Q' \in \mathfrak{g}^1$ is Maurer-Cartan, for every $v \in \mathfrak{g}^0/\mathfrak{h}^0$ we have:

$$R_{v,Q'}(\mathrm{ev}_{Q'}(v)) = 0 \in \mathfrak{g}^2/\mathfrak{h}^2.$$

We recall the way the result follows from these properties. Let C be a complement to

$$\ker(\overline{\partial + [Q,-]} : \mathfrak{g}^1/\mathfrak{h}^1 \to \mathfrak{g}^2/\mathfrak{h}^2) = \mathrm{im}(\overline{\partial + [Q,-]} : \mathfrak{g}^0/\mathfrak{h}^0 \to \mathfrak{g}^1/\mathfrak{h}^1)$$

in $\mathfrak{g}^1/\mathfrak{h}^1$.
First property b) implies that $R_{0,Q}$ restricted to C is an immersion at $0 \in C$ as C has trivial intersection with $\ker(\overline{\partial + [Q,-]} : \mathfrak{g}^1/\mathfrak{h}^1 \to \mathfrak{g}^2/\mathfrak{h}^2 = (D(R_{0,Q}))_0)$. By Lemma 3.6.3, there exists an open neighborhood O of $0 \in C$, an open neighborhood S of $0 \in V$ and a neighborhood $\mathcal{U}_2$ of $Q \in \mathfrak{g}^1$ such that $R_{v,Q'}\big|_O$ is an injective immersion for $Q' \in \mathcal{U}_2$ and $v \in S$, where we use the continuous dependence of R on the parameters $(v, Q') \in V \times \mathfrak{g}^1$.
Property a) implies that ev_Q intersects $O \subseteq C$ transversely in 0, as

$$\overline{\partial + [Q,-]} : \mathfrak{g}^0/\mathfrak{h}^0 \to \mathfrak{g}^1/\mathfrak{h}^1 = -(D(\mathrm{ev}_Q))_0,$$

and

$$\ker(\overline{\partial + [Q,-]} : \mathfrak{g}^1/\mathfrak{h}^1 \to \mathfrak{g}^2/\mathfrak{h}^2) = \mathrm{im}(\overline{\partial + [Q,-]} : \mathfrak{g}^0/\mathfrak{h}^0 \to \mathfrak{g}^1/\mathfrak{h}^1)$$

by the cohomological assumption. Therefore by Lemma 3.6.1, there exists a neighborhood $\mathcal{U}_1$ of $Q \in \mathfrak{g}^1$ such that for any $Q' \in \mathcal{U}_1$, there exists a $v \in S$ such that $\mathrm{ev}_{Q'}(v) \in O$.
By property c), for any Maurer-Cartan element $Q' \in \mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2$, and any $v \in V$ we have

$$R_{v,Q'}(\mathrm{ev}_{Q'}(v)) = 0.$$

By injectivity of $R_{v,Q'}$ restricted to O, combined with the fact that $\mathrm{ev}_{Q'}(v) \in O$, it follows that

$$\mathrm{ev}_{Q'}(v) = 0,$$

or equivalently, $(Q')^{\sigma_0(v)} \in \mathfrak{h}^1$. For the existence of the family of fixed points, apply the final statement of Lemma 3.6.1, again using that $R_{v,Q'}$ is injective restricted to O.
We define the maps.

i) Let $Q' \in \mathfrak{g}^1$. Then for $v \in V$, we set

$$\mathrm{ev}_{Q'}(v) = (Q')^{\sigma_0(v)} + \mathfrak{h}^1.$$

Then by assumption on the gauge action, the map depends continuously on Q' and smoothly on $v \in V$.

ii) Next, for $(v, Q') \in V \times \mathfrak{g}^1$, $\hat{Q} + \mathfrak{h}^1 \in \mathfrak{g}^1/\mathfrak{h}^1$ we set

$$R_{v,Q'}(\hat{Q} + \mathfrak{h}^1) = \partial(\hat{Q}) + [(Q')^{\sigma_0(v)} - \sigma_1((Q')^{\sigma_0(v)} + \mathfrak{h}^1), \sigma_1(\hat{Q} + \mathfrak{h}^1)] + \frac{1}{2}[\sigma_1(\hat{Q} + \mathfrak{h}^1), \sigma_1(\hat{Q} + \mathfrak{h}^1)] + \mathfrak{h}^2.$$

a) The properties regarding the value of $\mathrm{ev}_{Q'}$ hold by definition. For the differential, we compute for $v \in \mathfrak{g}^0/\mathfrak{h}^0 \cong T_0V$

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} \mathrm{ev}_Q(tv) &= \left.\frac{d}{dt}\right|_{t=0} (Q)^{t\sigma_0(v)} + \mathfrak{h}^1 \\
&= \left.\left[\sigma_0(v), Q^{t\sigma_0(v)}\right] - \partial(v)\right|_{t=0} + \mathfrak{h}^1 \\
&= [\sigma_0(v), Q] - \partial(v) + \mathfrak{h}^1 \\
&= -\overline{\partial + [Q, -]}(v).
\end{aligned}$$

b) The properties regarding the values and derivatives of $R_{v,Q'}$ are immediate.

c) Finally we compute for $Q' \in \mathfrak{g}^1$, $v \in V$, we compute. Let $X = (Q')^{\sigma_0(v)}$, and let $\overline{X} := X + \mathfrak{h}^1$.

$$\begin{aligned}
R_{v,Q'}(\mathrm{ev}_{Q'}(v)) =& \partial(X) + \left[X - \sigma_1(\overline{X}), \sigma_1(\overline{X})\right] + \frac{1}{2}\left[\sigma_1(\overline{X}), \sigma_1(\overline{X})\right] + \mathfrak{h}^2 \\
=& \partial(X) + \left[X - \sigma_1(\overline{X}), \sigma_1(\overline{X})\right] + \frac{1}{2}\left[\sigma_1(\overline{X}), \sigma_1(\overline{X})\right] \\
&+ \frac{1}{2}\left[X - \sigma_1(\overline{X}), X - \sigma_1(\overline{X})\right] + \mathfrak{h}^2 \\
=& \partial(X) + \frac{1}{2}\left[X - \sigma_1(\overline{X}) + \sigma_1(\overline{X}),\ X - \sigma_1(\overline{X}) + \sigma_1(\overline{X})\right] \\
=& \partial(X) + \frac{1}{2}[X, X] + \mathfrak{h}^2,
\end{aligned}$$

which vanishes if Q' is a Maurer-Cartan element. Here the second equality holds because $X - \sigma_1(X + \mathfrak{h}^1) \in \mathfrak{h}^1$, and $[\mathfrak{h}^1, \mathfrak{h}^1] \subseteq \mathfrak{h}^2$.

□

Remark 3.3.21. We start with a couple of remarks, highlighting the differences with the proof of Theorem 3.2.8.

i) The map ∂ does not appear in Theorem 3.2.8. As the input of Theorem 3.3.20 consists of a differential graded Lie algebra, and a Maurer-Cartan element Q to start with, when the differential is an inner derivation of the graded Lie algebra one can take $\partial = 0$, and look at Maurer-Cartan elements near Q rather than Maurer-Cartan elements of the *differential* graded Lie algebra $(\mathfrak{g}, [Q, -], [-, -])$ near $0 \in \mathfrak{g}^1$. One reason to do this is that the initial value problem for the gauge equation is homogeneous. As we will see in Section 3.5.3, this cannot be done if ∂ is not inner.

ii) The role of σ_0 and σ_1 in the case of Theorem 3.2.8 was to extend elements $v \in T_pM$ and $\rho \in A_p^* \otimes T_pM$ to constant sections of their respective vector bundles. This explains why the map $R_{v,Q'}$ was linear in that example: the pair $(\rho, [-, -])$ defined by a constant anchor and a zero bracket satisfies the axioms of a Lie algebroid, hence the second term is identically zero.

iii) Note that although σ_1 appears in the proof, it does not appear in the statement. It is however necessary: as $\mathfrak{h}$ is not an ideal, but only a graded Lie subalgebra, the quotients do not inherit a graded Lie algebra structure.

iv) The requirement that $[-, -] : \mathfrak{g}^1 \times \mathfrak{g}^1 \to \mathfrak{g}^2$ is continuous is needed to show that

$$R : U \times \mathfrak{g}^1 \to C^\infty\left(\mathfrak{g}^1/\mathfrak{h}^1, \mathfrak{g}^2/\mathfrak{h}^2\right)$$

$$(v, Q') \mapsto R_{v,Q'}$$

is continuous. This condition can be replaced by a different condition which is easier to check, and is satisfied in our applications. We give it in Lemma 3.3.22 below.

v) Marco Zambon pointed out that Theorem 3.3.20 has a deformation theoretic interpretation. The inclusion $i : (\mathfrak{h}, \partial, [-, -]) \hookrightarrow (\mathfrak{g}, \partial, [-, -])$ is a map of differential graded Lie algebras, hence induces a map on the level of Maurer-Cartan sets

$$i_{MC} : MC(\mathfrak{h}) \to MC(\mathfrak{g}),$$

and on the level of Maurer-Cartan varieties

$$\overline{i_{MC}} : MC(\mathfrak{h})/\mathfrak{h}^0 \to MC(\mathfrak{g})/\mathfrak{g}^0.$$

The cohomological assumption $H^1(\mathfrak{g}/\mathfrak{h}, \overline{\partial + [Q,-]}) = 0$ implies that the induce map on the degree 1 cohomology

$$H^1(i) : H^1(\mathfrak{h}, \partial + [Q,-]) \to H^1(\mathfrak{g}, \partial + [Q,-])$$

is surjective by the long exact sequence associated to the short exact sequence of cochain complexes

$$0 \longrightarrow (\mathfrak{h}, \partial + [Q,-]) \xrightarrow{i} (\mathfrak{g}, \partial + [Q,-]) \longrightarrow (\mathfrak{g}/\mathfrak{h}, \overline{\partial + [Q,-]}) \longrightarrow 0.$$

As $H^1(\mathfrak{h}, \partial + [Q,-])$ and $H^1(\mathfrak{g}, \partial + [Q,-])$ morally play the role of the tangent spaces to the respective Maurer-Cartan varieties at Q with $H^1(i)$ playing the role of the differential of i, Theorem 3.3.20 turns a stronger version of surjectivity of $H^1(i)$ into (local) surjectivity of $\overline{i_{MC}}$.

vi) It would be interesting to know if the requirement $H^1(\mathfrak{g}/\mathfrak{h}, \overline{\partial + [Q,-]}) = 0$ can be weakened to the surjectivity of $H^1(i)$.

vii) Theorem 3.3.20 only yields a sufficient condition, and not a necessary one. In some places of the text we comment on this.

viii) By modifying the maps ev and R, we can generalize this result to the case where $\mathfrak{g}$ is an L_∞-algebra, and $\mathfrak{h}$ a strict L_∞-subalgebra. We will elaborate on this in future work.

ix) If there exists a subspace $K \subseteq \ker(\overline{\partial + [Q,-]} : \mathfrak{g}^1/\mathfrak{h}^1 \to \mathfrak{g}^2/\mathfrak{h}^2)$ such that for any Maurer-Cartan element $Q' \in MC(\mathfrak{g})$, $Q' + \mathfrak{h}^1 \in K$, we can replace $H^1(\mathfrak{g}/\mathfrak{h}, \overline{\partial + [Q,-]})$ in the hypothesis by

$$H^1_{red} := \frac{K}{\operatorname{im}(\overline{\partial + [Q,-]} : \mathfrak{g}^0/\mathfrak{h}^0 \to K)},$$

and the conclusion remains true. Observe that the space we quotient by is well-defined: As by assumption Q is a Maurer-Cartan element, so is $Q^{t\sigma_0(v)}$ for all $t \in \mathbb{R}, v \in \mathfrak{g}^0/\mathfrak{h}^0$, and the differential $\partial + [Q,-]$ is the t-derivative at $t = 0$.

x) In applications, we will often only specify $\mathfrak{h}$ in degrees 0, 1 and 2. To complete this into a differential graded Lie subalgebra, we can take $\mathfrak{h}^i = \mathfrak{g}^i$ for $i > 2$.

There is a simpler condition than the continuity of $[-,-] : \mathfrak{g}^1 \times \mathfrak{g}^1 \to \mathfrak{g}^2$ which guarantees the continuity of

$$R : U \times \mathfrak{g}^1 \to C^\infty\left(\mathfrak{g}^1/\mathfrak{h}^1, \mathfrak{g}^2/\mathfrak{h}^2\right)$$

where U is as in assumptions 3.3.17iv).

Lemma 3.3.22. *Assume that there exists a subspace $F \subseteq \mathfrak{g}^1$ such that*

- *F has finite codimension in $\mathfrak{g}^1$,*
- *$p_F : \mathfrak{g}^1 \to \mathfrak{g}^1/F$ is continuous,*
- *The bracket $[-,-] : \mathfrak{g}^1 \times \mathfrak{g}^1 \to \mathfrak{g}^2/\mathfrak{h}^2$ factors through $\overline{[-,-]} : \mathfrak{g}^1/F \times \mathfrak{g}^1/F \to \mathfrak{g}^2/\mathfrak{h}^2$.*

Then the map $R : U \times \mathfrak{g}^1 \to C^\infty\left(\mathfrak{g}^1/\mathfrak{h}^1, \mathfrak{g}^2/\mathfrak{h}^2\right)$ is continuous, when the right hand side carries the C^1-topology.

Proof. Note first that because the assignment $U \times \mathfrak{g}^1 \ni (v, Q') \mapsto (Q')^{\sigma_0(v)}$ is continuous, it is sufficient to show that the restriction of R to $\{0\} \times \mathfrak{g}^1$ is continuous. Further, while the map R is not linear in $\mathfrak{g}^1$, it is not very far from it, as it is affine. So continuity of R is equivalent to the continuity of the map

$$R - R_{0,0} : \{0\} \times \mathfrak{g}^1 \to C^\infty(\mathfrak{g}^1/\mathfrak{h}^1, \mathfrak{g}^2/\mathfrak{h}^2).$$

This map takes values in $\mathrm{Hom}(\mathfrak{g}^1/\mathfrak{h}^1, \mathfrak{g}^2/\mathfrak{h}^2)$, the *linear* maps, and is given by

$$\begin{aligned}(R_{0,Q'} - R_{0,0}) &= [Q' - \sigma_1(Q' + \mathfrak{h}^1), \sigma_1(-)] + \mathfrak{h}^2 \\ &= \overline{[p_F(Q' - \sigma_1(Q' + \mathfrak{h}^1), p_F(\sigma_1(-))]} \in \mathfrak{g}^2/\mathfrak{h}^2,\end{aligned}$$

which is the composition of a continuous linear map p_F with a linear map between finite-dimensional vector spaces, hence continuous. ☐

Remark 3.3.23. A way to think about Lemma 3.3.22 is as follows: given a manifold M, the graded Lie algebra $\mathfrak{g} = \mathfrak{X}^\bullet(M)[1]$ of multivector fields and a point $p \in M$ we can take $\mathfrak{h} := I_p\mathfrak{X}^\bullet(M)[1]$. Then, $\mathfrak{g}^i/\mathfrak{h}^i \cong \wedge^{i+1}T_pM$, and the quotient map is simply the evaluation at $p \in M$. To compute the value of the Schouten-Nijenhuis bracket $[\pi_1, \pi_2]_{SN}$ of bivector fields $\pi_1, \pi_2 \in \mathfrak{X}^2(M)$ in p however, it is not sufficient to know the values of π_1, π_2 at $p \in M$: we also need to know the value of their first derivatives. In other words, we need to know the equivalence class of π_1, π_2 mod $I_p^2\mathfrak{X}^2(M)$, which could be taken as F.

3.4 Higher order fixed points of Lie (n-)algebroids

In this section we give some applications of Theorem 3.3.20. We first apply it to obtain a stability result for higher order fixed points of Lie algebroids (Theorem 3.4.11), which we then show to be equivalent to Theorem 1.3 of [DW06].

We then apply it to obtain a stability result for (higher order) fixed points of Lie n-algebroids (Theorems 3.4.23 and 3.4.30).

The results for Lie n-algebroids can then be applied to singular foliations, and we obtain a stability result (Proposition 3.4.42) and a formal rigidity result (Corollary 3.4.46) for linear singular foliations.

3.4.1 Higher order fixed points of Lie algebroids

Let (A, Q) be a Lie algebroid over the manifold M. The first application will be Theorem 1.3 of [DW06].
This theorem yields a cohomological criterium for stability of a *higher order* fixed point of a Lie algebroid, which we define below.

3.4.1.1 The ingredients

In this section we show that we are in the setting of assumptions 3.3.17.

i) As we still deal with Lie algebroid structures, we take the graded Lie algebra
$$(\mathfrak{g}_{LA} = \mathfrak{X}(A[1]), 0, [-,-]).$$

ii) We now define the graded Lie subalgebra of $\mathfrak{g}_{LA}$ which contains the Lie algebroid structures with a higher order fixed point $p \in M$.

Definition 3.4.1. Let $p \in M$, $j \geq -1, k \geq 0$ and let A be a vector bundle. Define

$$\mathfrak{X}^j_{p,k}(A[1]) := \{\delta \in \mathfrak{X}^j(A[1]) \,|\delta(C^\infty(M)) \subseteq I_p^k\Gamma(S^j(A^*[-1])),$$
$$\delta(\Gamma(A^*[-1])) \subseteq I_p^{k-1}\Gamma(S^{j+1}(A^*[-1]))\},$$

where $I_p \subseteq C^\infty(M)$ is the ideal of functions vanishing at p.

These subspaces behave well with respect to the graded commutator of vector fields, extending Lemma 3.3.9. We leave the proof to the reader.

Lemma 3.4.2. *For $p \in M$, $j_1, j_2 \geq 0, k_1, k_2 \geq 0$, we have*

$$[\mathfrak{X}^{j_1}_{p,k_1}(A[1]), \mathfrak{X}^{j_2}_{p,k_2}(A[1])] \subseteq \mathfrak{X}^{j_1+j_2}_{p,k_1+k_2-1}.$$

Remark 3.4.3. As the numbers $k_1, k_2, k_1 + k_2 - 1$ represent the order to which elements of the respective spaces vanish at p in a certain sense, Lemma 3.4.2 is not too surprising. It is an extension of the fact that given vector fields $X, Y \in \mathfrak{X}(M)$ vanishing to order $K_1, K_2 \geq 0$ in a point respectively, their Lie bracket $[X, Y]$ vanishes to order $K_1 + K_2 - 1$.

Corollary 3.4.4. *For $p \in M$,*

$$\left(\mathfrak{g}_{LA}(p,k) := \bigoplus_{j=0}^{\mathrm{rk}(A)} \mathfrak{X}^j_{p,j(k-1)+1}(A[1]), 0, [-,-]\right)$$

is a graded Lie subalgebra of $(\mathfrak{X}(A[1]), 0, [-,-])$.

We now have a concise way to define fixed points of higher order.

Definition 3.4.5. Let (A, Q) be a Lie algebroid over M, $p \in M$, $k \geq 1$. We say p is a *fixed point of order k of Q* if $Q \in \mathfrak{X}^1_{p,k}(A[1])$.

Remark 3.4.6.

- In classical terms, this means that $\rho(x) \in I_p^k \mathfrak{X}(M)$, $[x, y] \in I_p^{k-1}\Gamma(A)$ for every $x, y \in \Gamma(A)$, which is the same assumption as in [DW06].
- The requirement on the bracket is necessary: one way to see this is by noting that without a requirement on the "vertical part" of the vector fields in $\mathfrak{X}^j_{p,s}(A[1])$ for $j \geq 0, s \geq 0$, Lemma 3.4.2 would not be true.
- In classical terms, another motivation for this assumption arises using the structure equations of a Lie algebroid. As for any $x, y \in \Gamma(A)$,

$$\rho([x, y]) = [\rho(x), \rho(y)],$$

  the right hand side lies in $I_p^{2k-1}\mathfrak{X}(M)$, if $\rho(x), \rho(y) \in I_p^k \mathfrak{X}(M)$. One way to make sure that the same holds for the left hand side, is to require that $[x, y] \in I_p^{k-1}\Gamma(A)$.

We therefore obtain a graded Lie subalgebra $\mathfrak{g}_{LA}(p,k) \subseteq \mathfrak{g}_{LA}$ corresponding to Lie algebroid structures for which $p \in M$ is a fixed point of order k. The cochain spaces of the relevant complex will consist of $\mathfrak{g}_{LA}/\mathfrak{g}_{LA}(p,k)$. We will give a description of these vector spaces, proving that the quotients $\mathfrak{g}^i_{LA}/\mathfrak{g}^i_{LA}(p,k)$ are finite-dimensional for $i = 0, 1, 2$. Note that Lemma 3.3.15 extends to $\mathfrak{g}_{LA}(p,k)$:

Lemma 3.4.7. *For $j, k \geq 0$, let $r := j(k-1)$. Let $V := A[1]$, and $V^* = A^*[-1]$. There is a short exact sequence of $C^\infty(M)$-modules:*

$$0 \longrightarrow I_p^r \Gamma(S^{j+1}(V^*) \otimes V) \xrightarrow{\iota} \mathfrak{X}^j_{p,r+1}(V) \xrightarrow{\sigma} I_p^{r+1}\Gamma(S^j(V^*) \otimes TM) \longrightarrow 0 \ ,$$

where tensor products are over $C^\infty(M)$.

Consequently, the cochain spaces $\mathfrak{g}_{LA}/\mathfrak{g}_{LA}(p,k)$ sit inside a short exact sequence:

Corollary 3.4.8. *Using notation from Lemma 3.4.7, there is a short exact sequence*

$$0 \longrightarrow J_p^{r-1}(S^{j+1}(V^*)\otimes V) \xrightarrow{\bar{\iota}} \mathfrak{X}^j(V)/\mathfrak{X}^j_{p,r+1}(V) \xrightarrow{\bar{\sigma}} J_p^r(S^j(V^*)\otimes TM) \longrightarrow 0,$$

where for a vector bundle E and a positive integer l,

$$J_p^l(E) := \Gamma(E)/I_p^{l+1}\Gamma(E)$$

are the l-jets of E at p. In particular, the cochain spaces are finite dimensional vector spaces.

The differential however, does not restrict to the outer parts of the sequence.

Remark 3.4.9. For $k=1$, we have $r=0$, and we are back in the setup of Section 3.2.

iii) By restricting the graded Lie algebra to a sufficiently small coordinate neighborhood of the point $p\in M$ over which A trivializes, we can choose splittings $\sigma_i : \mathfrak{g}^i_{LA}/\mathfrak{g}^i_{LA}(p,k) \to \mathfrak{g}^i_{LA}$ for $i=0,1$ by lifting jets to polynomial sections.

iv) Now pick a Lie algebroid structure Q such that $p\in M$ is a fixed point of order k, i.e. a Maurer-Cartan element $Q\in\mathfrak{g}^1_{LA}(p,k)$.

We now check that the data above satisfies assumptions 3.3.17.

a) As each of the degrees of $\mathfrak{g}_{LA}$ are the sections of some vector bundle we take various C^s-topologies.

   - On $\mathfrak{g}^0_{LA}$, take the C^∞-topology,
   - On $\mathfrak{g}^1_{LA}$, take the C^{2k-1}-topology,
   - On $\mathfrak{g}^2_{LA}$, take the C^{2k-2}-topology.

   These choices make the projections continuous.

b) The differential is identically zero, hence is continuous.

c) This follows from a local computation, where it is crucial that for $X, Y \in \mathfrak{g}^1_{LA}$, the $(2k-2)$-jet of $[X, Y]$ depends bilinearly on the $(2k-1)$-jets of X and Y.

d) As the gauge action for $\mathfrak{g}_{LA}$ is given by the flow of some vector field, the choice of the splittings above implies that it is defined for any $v \in \mathfrak{g}^0_{LA}/\mathfrak{g}^0_{LA}(p, k)$.
Finally,

Lemma 3.4.10. *The gauge action* $\mathfrak{g}^0_{LA}/\mathfrak{g}^0_{LA}(p, k) \times \mathfrak{g}^1_{LA} \to \mathfrak{g}^1_{LA}$ *is continuous.*

Proof. We need to show that for $(v, Q_1) \in \mathfrak{g}^0_{LA}/\mathfrak{g}^0_{LA}(p, k) \times \mathfrak{g}^1_{LA}$, any compact set $K \subseteq M$, and $\epsilon > 0$, there is a compact set $K' \subseteq M$, $\epsilon' > 0$ and $\delta > 0$ with the property that if

$$\|Q_1 - Q_2\|_{K',2k-1} < \epsilon', \|v - w\| < \delta,$$

we have

$$\|Q_1^{\sigma_0(v)} - Q_2^{\sigma_0(w)}\|_{K,2k-1} < \epsilon,$$

where $\|\cdot\|_{K,2k-1}$ denotes the C^{2k-1}-seminorm associated to the compact set K.
Note that

$$\begin{aligned}\|Q_1^{\sigma_0(v)} - Q_2^{\sigma_0(w)}\|_{K,2k-1} \leq &\|Q_1^{\sigma_0(v)} - Q_1^{\sigma_0(w)}\|_{K,2k-1} \\ &+ \|(Q_1 - Q_2)^{\sigma_0(w)}\|_{K,2k-1}.\end{aligned}$$

By uniform continuity of Q_1 restricted to K, there exists $\delta > 0$ such that the first term is at most $\frac{\epsilon}{2}$ if $\|v - w\| < \delta$. Recall that the gauge action by $\sigma_0(u)$ is given by some vector bundle automorphism of $\mathfrak{g}^1_{LA}$, and vector bundle automorphisms induce continuous maps on the space of sections. Hence there exists some constant $C > 0$, such that for $K' = \phi(\overline{B_v(\delta)} \times K)$

$$\|(Q_1 - Q_2)^{\sigma_0(w)}\|_{K,2k-1} \leq C\|Q_1 - Q_2\|_{K',2k-1}.$$

Here $\phi : \mathfrak{g}^0_{LA}/\mathfrak{g}^0_{LA}(p, k) \times K \to M$ is defined for $(w, x) \in \mathfrak{g}^0_{LA}/\mathfrak{g}^0_{LA}(p, k) \times K$ by

$$\phi(w, x) = \phi_w(x).$$

Here ϕ_w is the time-1 flow on M of the symbol of the element $\sigma_0(w)$, using that ϕ is continuous.
Setting $\epsilon' = \frac{\epsilon}{2C}$ then yields the result. ☐

e) Lemma 3.3.19 implies that the gauge action preserves Maurer-Cartan elements.

3.4.1.2 Applying the main theorem

Now that all assumptions are satisfied, taking $\mathfrak{g}_{LA} = \mathfrak{X}(A[1])$, $\mathfrak{g}_{LA}(p,k) = \bigoplus_{i=0}^{\mathrm{rk}(A)} \mathfrak{X}^i_{p,i(k-1)+1}(A[1])$, the main theorem implies:

Theorem 3.4.11. *Let (A,Q) be a Lie algebroid over M. Let $p \in M$ be a fixed point of order $k \geq 1$, that is, $Q \in \mathfrak{g}_{LA}(p,k)$. Assume that*

$$H^1(\mathfrak{g}_{LA}/\mathfrak{g}_{LA}(p,k), \overline{[Q,-]}) = 0.$$

Then for every open neighborhood U of $p \in M$, there exists a C^{2k-1}-neighborhood $\mathcal{U}$ of Q such that for any Lie algebroid structure $Q' \in \mathcal{U}$ there is a family I in U of fixed points of order k of Q' parametrized by an open neighborhood of

$$0 \in H^0(\mathfrak{g}_{LA}/\mathfrak{g}_{LA}(p,k), \overline{0^Q}).$$

3.4.1.3 Equivalence with the Dufour-Wade stability theorem for Lie algebroids

In this section we compare Theorem 3.4.11 with Theorem 1.3 of [DW06]. We will show that Theorem 3.4.11 is equivalent to Theorem 1.3 of [DW06]. The conclusions of the theorems are equivalent so we will show that the cohomological assumptions are also equivalent. The theorem was originally proven in [DW06] using a special kind of multivector fields on A^* rather than vector fields on $A[1]$, and most notably, the differentials are not the same.
The first difference can be quite easily explained: the graded Lie algebra multivector fields on A^* which are fiber-wise linear [CM08, Section 4.9] are canonically isomorphic to the graded Lie algebra of multiderivations on A, and the same holds for vector fields on $A[1]$ [CM08, Section 2.5].
The second difference is that we work with a quotient of the vector fields on $A[1]$, whereas in [DW06] the authors work with representatives of the classes in the quotients, namely the multivector fields on A^*, which are fiber-wise linear, and polynomial in coordinates on $M = V$. However, even when making these identifications, the differential

$$\overline{[Q,-]} : \mathfrak{g}_{LA}/\mathfrak{g}_{LA}(p,k) \to \mathfrak{g}_{LA}/\mathfrak{g}_{LA}(p,k)$$

and the differential of [DW06] on the complexes do not agree. We want to show that the vanishing of the respective cohomologies are equivalent conditions. To see this, we first do an intermediate step, which will double as a more convenient way to check the cohomological hypothesis of Theorem 3.4.11.
The complex

$$\mathfrak{X}^0(A[1])/\mathfrak{X}^0_{p,1}(A[1]) \xrightarrow{\overline{[Q,-]}} \mathfrak{X}^1(A[1])/\mathfrak{X}^1_{p,k}(A[1]) \xrightarrow{\overline{[Q,-]}} \mathfrak{X}^2(A[1])/\mathfrak{X}^2_{p,2k-1}(A[1])$$

has a finite descending filtration. Indeed: let for $t = 0, \dots, k$ (omitting the $A[1]$ for convenience):

$$F^t(\mathfrak{X}^0/\mathfrak{X}^0_{p,1}) := \begin{cases} \mathfrak{X}^0/\mathfrak{X}^0_{p,1} & t = 0, \dots, k-1 \\ 0 & t = k \end{cases},$$

$$F^t(\mathfrak{X}^1/\mathfrak{X}^1_{p,k}) := \mathfrak{X}^1_{p,t}/\mathfrak{X}^1_{p,k},$$

$$F^t(\mathfrak{X}^2/\mathfrak{X}^2_{p,2k-1}) := \mathfrak{X}^2_{p,k-1+t}/\mathfrak{X}^2_{p,2k-1}.$$

The differentials preserve this filtration, so we obtain k complexes on the graded quotient: for $t = 0, \dots, k-2$ we get

$$0 \longrightarrow \mathfrak{X}^1_{p,t}/\mathfrak{X}^1_{p,t+1} \xrightarrow{gr(\overline{[Q,-]})} \mathfrak{X}^2_{p,t+k-1}/\mathfrak{X}^2_{p,t+k} , \tag{3.9}$$

and for $t = k-1$ we have

$$\mathfrak{X}^0/\mathfrak{X}^0_{p,1} \xrightarrow{gr(\overline{[Q,-]})} \mathfrak{X}^1_{p,k-1}/\mathfrak{X}^1_{p,k} \xrightarrow{gr(\overline{[Q,-]})} \mathfrak{X}^2_{p,2k-2}/\mathfrak{X}^2_{p,2k-1}. \tag{3.10}$$

Denote the corresponding cohomology groups by

$$gr_t H^1(\mathfrak{g}_{LA}/\mathfrak{g}_{LA}(p,k), \overline{[Q,-]}) \tag{3.11}$$

for $t = 0, \dots, k-1$. The vanishing of these cohomologies is equivalent to the vanishing of $H^1(\mathfrak{g}_{LA}/\mathfrak{g}_{LA}(p,k), \overline{0}^Q)$.

Proposition 3.4.12.

$$H^1(\mathfrak{g}_{LA}/\mathfrak{g}_{LA}(p,k), \overline{[Q,-]}) = 0 \iff gr_t H^1(\mathfrak{g}_{LA}/\mathfrak{g}_{LA}(p,k), \overline{[Q,-]}) = 0$$

for $t = 0, \dots, k-1$.

Proof. " $\implies$ " The key observation is that we can linear splittings of the sequences

$$0 \longrightarrow \mathfrak{X}^i_{p,t+1} \longrightarrow \mathfrak{X}^i_{p,t} \longrightarrow \mathfrak{X}^i_{p,t}/\mathfrak{X}^i_{p,t+1} \longrightarrow 0$$

for $i = 1, t = 0, \dots, k-1$, and $i = 2, t = k-1, \dots, 2k-2$. This gives rise to (filtered) linear isomorphisms

$$\mathfrak{X}^i_{p,(i-1)(k-1)}/\mathfrak{X}^i_{p,k+(i-1)(k-1)} \cong \bigoplus_{t=0}^{k-1} \mathfrak{X}^i_{p,t+(i-1)(k-1)}/\mathfrak{X}^i_{p,t+1+(i-1)(k-1)}$$

for $i = 1, 2$. Decomposing the differential with respect to this isomorphism it is block upper triangular, with the block diagonal being precisely $gr(\overline{[Q,-]})$.
$'' \Longleftarrow ''$ Let $\delta + \mathfrak{X}^1_{p,k} \in \mathfrak{X}^1/\mathfrak{X}^1_{p,k}$ such that $[Q,\delta] \in \mathfrak{X}^2_{p,2k-1}$. Then in particular $[Q,\delta] \in \mathfrak{X}^2_{p,k} \supset \mathfrak{X}^2_{p,2k-1}$. As

$$gr_0 H^1(\mathfrak{g}_{LA}/\mathfrak{g}_{LA}(p,k), \overline{[Q,-]}) = 0,$$

we find that $\delta \in \mathfrak{X}_{p,1}$, using (3.9) for $t = 0$. Applying this reasoning inductively, we eventually find that $\delta \in \mathfrak{X}_{p,k-1}$.
Finally, the assumption

$$gr_{k-1} H^1(\mathfrak{g}_{LA}/\mathfrak{g}_{LA}(p,k), \overline{[Q,-]}) = 0,$$

implies that there exists $X \in \mathfrak{X}^0$ such that $[Q,X] - \delta \in \mathfrak{X}^1_{p,k}$ using (3.10), concluding the proof. □

Now we can make a direct connection with [DW06]: the cohomology groups appearing in the theorem are exactly $gr_t H^1(\mathfrak{g}_{LA}/\mathfrak{g}_{LA}(p,k), \overline{[Q,-]})$.

Proposition 3.4.13. *Let $(A = \mathfrak{g} \times M, Q)$ be a Lie algebroid over the vector space $M = V$ such that $p = 0 \in M$ is a fixed point of order k. Let Π_Q denote the corresponding fiberwise linear Poisson structure on A^*. Then*

$$H^{2,t}_{lin}(\Pi_Q^{(k)}) \cong gr_t H^1(\mathfrak{g}_{LA}/\mathfrak{g}_{LA}(p,k), \overline{[Q,-]})$$

for $t = 0, \dots, k-1$. Here the left hand side denotes the cohomology as defined in [DW06], where $\Pi_Q^{(k)}$ denotes the k-th order Taylor expansion of Π_Q around 0_p. The right hand side denotes the graded cohomology of the filtered complex as defined above (3.11)*.*

Proof. As shown in [CM08], there is an $C^\infty(M)$-linear isomorphism of graded Lie algebras

$$\Phi : \mathfrak{X}^\bullet_{lin}(A^*)[1] \to \mathfrak{X}(A[1]).$$

Let $p \in M$ denote the origin. On the left hand side, we can define subspaces corresponding to the order of vanishing at $0_p \in A^*$: For $q, t \geq 0$, let

$$\mathfrak{X}^q_{lin}(A^*)_{p,t} := I^t_{0_p} \mathfrak{X}^q(A^*) \cap \mathfrak{X}^q_{lin}(A^*),$$

where $I_{0_p} \subseteq C^\infty(A^*)$ is the ideal of functions on A^* vanishing at $0_p \in A^*$. For brevity, we omit the argument A^* in the following. It is straightforward to verify that

$$[(\mathfrak{X}^{q_1}_{lin})_{p,k_1}, (\mathfrak{X}^{q_2}_{lin})_{p,k_2}] \subseteq (\mathfrak{X}^{q_1+q_2-1}_{lin})_{p,k_1+k_2-1},$$

and $\Phi((\mathfrak{X}^q_{lin})_{p,t}) = \mathfrak{X}^{q-1}_{p,t}(A[1])$. In particular, the assumption that p is a fixed point of order k in terms of Π_Q means that $\Pi_Q \in \mathfrak{X}^2(A^*)_{p,k}$. This implies that the differential $[\Pi_Q, -]$ induces a differential on the quotients

$$(\mathfrak{X}^1_{lin})/(\mathfrak{X}^1_{lin})_{p,1} \xrightarrow{\overline{[\Pi_Q,-]}} (\mathfrak{X}^2_{lin})/(\mathfrak{X}^2_{lin})_{p,k} \xrightarrow{\overline{[\Pi_Q,-]}} (\mathfrak{X}^3_{lin})/(\mathfrak{X}^3_{lin})_{p,2k-1}.$$

Again the order of vanishing gives rise to a filtered complex, and Φ descends to a filtered isomorphism

$$\begin{array}{ccccc} (\mathfrak{X}^1_{lin})/(\mathfrak{X}^1_{lin})_{p,1} & \xrightarrow{\overline{[\Pi_Q,-]}} & (\mathfrak{X}^2_{lin})/(\mathfrak{X}^2_{lin})_{p,k} & \xrightarrow{\overline{[\Pi_Q,-]}} & (\mathfrak{X}^3_{lin})/(\mathfrak{X}^3_{lin})_{p,2k-1} \\ \downarrow{\overline{\Phi}} & & \downarrow{\overline{\Phi}} & & \downarrow{\overline{\Phi}} \\ \mathfrak{X}^0(A[1])/\mathfrak{X}^0(A[1])_{p,1} & \xrightarrow{\overline{[Q,-]}} & \mathfrak{X}^1(A[1])/\mathfrak{X}^1(A[1])_{p,k} & \xrightarrow{\overline{[Q,-]}} & \mathfrak{X}^2(A[1])/\mathfrak{X}^2(A[1])_{p,2k-1} \end{array},$$

giving rise to an isomorphism of the corresponding graded quotients. It remains to be shown that $H^{2,s}_{lin}(\Pi^{(k)})$ is isomorphic to the cohomology of the corresponding graded quotient

$$0 \longrightarrow (X^2_{lin})_{p,s}/(\mathfrak{X}^2_{lin})_{p,s+1} \xrightarrow{gr(\overline{[\Pi_Q,-]})} (\mathfrak{X}^3_{lin})_{p,s+k-1}/(\mathfrak{X}^3_{lin})_{p+s+k}$$

for $s = 0, \ldots, k-2$, and

$$(\mathfrak{X}^1_{lin})/(\mathfrak{X}^1_{lin})_{p,1} \xrightarrow{gr(\overline{[\Pi_Q,-]})} (\mathfrak{X}^2_{lin})_{p,k-1}/(\mathfrak{X}^2_{lin})_{p,k} \xrightarrow{gr(\overline{[\Pi_Q,-]})} (\mathfrak{X}^3_{lin})_{p,2k-2}/(X^3_{lin})_{p,2k-1}$$

for $s = k-1$. However, there is already an isomorphism on the level of cochain complexes: by identifying $\mathcal{V}^{(s)}_{i,lin}(T_{0_p}A^*)$ with multivector fields on A^* with degree s polynomial coefficients in $M = V$, we get a complete set of representatives of $\mathfrak{X}^i_{lin}(A^*)_{p,s}/\mathfrak{X}^i_{lin}(A^*)_{p,s+1}$. Moreover, this also intertwines the differentials: given any homogeneous linear multivector field $A \in \mathcal{V}^{(s)}_{i,lin}$, this amounts to checking that

$$[\Pi_Q, A] - [\Pi^{(k)}_Q, A] \in \mathfrak{X}^i_{lin}(A^*)_{p,k+s}.$$

But this is immediate, as $\Pi_Q - \Pi^{(k)}_Q \in \mathfrak{X}^2_{lin}(A^*)_{p,k+1}$. □

3.4.2 Fixed points of Lie n-algebroids

In this section we apply the main theorem to Lie n-algebroids and obtain a result analogous to Theorem 3.4.11. In classical terms, Lie n-algebroids can be described as a graded vector bundle with an anchor, and a collection of higher brackets on the space of its sections. We will first look at the case when the anchor vanishes in a point $p \in M$, and interpret Theorem 3.3.20 in this setting.

3.4.2.1 Lie n-algebroids

We start with the definition of a Lie n-algebroid. Just as for Lie algebroids there are several ways to characterize Lie n-algebroids. We choose the graded geometric point of view.

Definition 3.4.14. Let M be a smooth manifold. A *Lie n-algebroid* over M is a pair (E, Q), where

i) $E = \bigoplus_{i=1}^n E_i[i-1]$ is a non-positively graded vector bundle,

ii) $Q : \Gamma(S(E^*[-1])) \to \Gamma(S(E^*[-1]))$ is a degree 1 $\mathbb{R}$-linear derivation,

satisfying

a) $Q^2 = \frac{1}{2}[Q, Q] = 0$.

We denote by

$$C^\infty(E[1], Q) := (\Gamma(S(E^*[-1])), Q)$$

the differential graded commutative algebra of *smooth functions* on $E[1]$, and by

$$(\mathfrak{X}(E[1]), [Q, -], [-, -]) := (\mathrm{Der}_{\mathbb{R}}(C^\infty(E[1])), [Q, -], [-, -])$$

the differential graded Lie algebra of *vector fields* on $E[1]$.

Remark 3.4.15.

i) $C^\infty(E[1])$ is bigraded as an algebra: the grading coming from the grading on E, and the grading coming from the symmetric power of $E^*[-1]$, which we will refer to as the *weight*.

ii) The bigrading on $C^\infty(E[1])$ induces a bigrading on $\mathfrak{X}(E[1])$. The grading coming from the grading on E will be called the *degree* of a vector field, while the grading coming from the weight will be called the *arity*.

iii) We do not require Q to respect the bigrading, merely that Q has degree 1. So Q can be decomposed as $Q = \sum_{i=0}^n Q^{(i)}$, where $Q^{(i)}$ raises the symmetric power by i.

iv) Due to T. Voronov [Vor10], the data of a Lie n-algebroid can be described equivalently in terms of an anchor $\rho : \Gamma(E_1) \to \mathfrak{X}(M)$ and a collection of multibrackets

$$\ell_k : S^k(E[1]) \to E[1]$$

for $k \geq 1$ of degree 1 satisfying quadratic identities.

v) When E is concentrated in degree 0, this reduces to the definition of a Lie algebroid.

vi) $\mathfrak{X}(E[1])$ is a graded Lie algebra whose Maurer-Cartan elements are precisely the Lie n-algebroid structures.

Although the functions and vector fields on $E[1]$ are more complicated to describe than in the Lie algebroid setting, the analogue of Lemma 3.3.15 still holds:

Proposition 3.4.16. *There is a short exact sequence of graded $C^\infty(M)$-modules:*

$$0 \longrightarrow C^\infty(E[1]) \otimes E[1] \xrightarrow{\iota} \mathfrak{X}(E[1]) \xrightarrow{\sigma} C^\infty(E[1]) \otimes \mathfrak{X}(M) \longrightarrow 0 \ ,$$

where tensor products are over $C^\infty(M)$, ι and σ are the contraction and restriction to $C^\infty(M)$ respectively. Moreover, any choice of connections on the E_i give rise to a splitting of the sequence.

Note however that there is a difference from the Lie algebroid case: the module $C^\infty(E[1])$ is not finitely generated as $C^\infty(M)$-module. Consequently, there is no (finite-dimensional) vector bundle F such that $C^\infty(E[1]) = \Gamma(F)$. Not all hope is lost however: the sequence above restricts to every degree, and every degree is finitely generated. So vector fields of a given degree are isomorphic to the sections of some vector bundle. This allows us to make sense of vector fields of given degree being C^p-close for some p.

Remark 3.4.17. Before we go on to apply Theorem 3.3.20, we make an observation, pointed out in [LGLS20]: For any Lie n-algebroid (E, Q), $(E_1, Q^{(1)})$ is an almost Lie algebroid, which makes sense as the component $Q^{(1)}$ preserves the functions on $E[1]$:

$$Q^{(1)}(\Gamma(S(E_1^*[-1]))) \subseteq \Gamma(S(E_1^*[-1])).$$

If we now define a fixed point of Q to be a fixed point of $Q^{(1)}$, Theorem 3.2.8 in combination with remark 3.2.9 now yields a stability criterium for fixed points of Lie n-algebroids.
A Lie n-algebroid however contains more data than just the underlying almost Lie algebroid structure on its degree -1 part. Therefore, if we want to know if a fixed point is stable for nearby Lie n-algebroid structures, we might be able to do better, taking into account properties of other components of Q. For this Theorem 3.3.20 will be useful.

3.4.2.2 The ingredients

In this section we show that we are in the setting of assumptions 3.3.17.

i) We first define the graded Lie algebra where things will take place. There is a slight difference with Lie algebroids here: while for Lie algebroids, the notion of arity and degree of vector fields coincide, that is not the case here. In particular, this implies that degree zero vector fields do not only consist of covariant differential operators on the graded vector bundle E, but also contain $C^\infty(M)$-linear maps $X : \Gamma(E_2^*[-2]) \to \Gamma(S^2(E_1^*[-1]))$ for example. The induced gauge action on a vector field by this element is nothing but

$$\mathfrak{X}^1(E[1]) \ni Q \mapsto Q + [X, Q] \in \mathfrak{X}^1(E[1]).$$

In particular, if Q is arity-homogeneous, Q gauge transformed by X will not be. Moreover, as X is $C^\infty(M)$-linear, it induces the identity on M, so it is not relevant for moving the fixed point around. We therefore define

$$\mathfrak{g}^i_{LnA} := \begin{cases} {}^0\mathfrak{X}^0(E[1]) & i = 0, \\ \mathfrak{X}^i(E[1]) & i > 0, \end{cases} \tag{3.12}$$

where ${}^0\mathfrak{X}^0(E[1])$ denotes the set of vector fields with degree and arity 0. We take the differential $\partial = 0$.

ii) Although the general algebraic framework does not allow such a uniform description as for Lie algebroids, given a fixed point $p \in M$ of order $(1, l)$ for $l \geq 0$ which we define below, we can still associate a graded Lie subalgebra $\mathfrak{g}_{LnA}(p, (1, l))$ of $\mathfrak{g}_{LnA}$ to it, such that its Maurer-Cartan elements are those Lie n-algebroid structures which have $p \in M$ a fixed point of order $(1, l)$. As the only relevant degrees to Theorem 3.3.20 are 0, 1 and 2, we only specify the Lie subalgebra in these degrees. The definition can be extended by setting $\mathfrak{g}^{\geq 3}_{LnA}(p, (1, l)) = \mathfrak{g}^{\geq 3}_{LnA}$ to obtain a Lie subalgebra.

Definition 3.4.18. Let $E = \bigoplus_{i=1}^n E_i[i-1]$ be a graded vector bundle over M, $p \in M$ and let $l \geq 0$ be an integer. Define

$$\mathfrak{g}^0_{LnA}(p,(1,l)) := \{\delta \in {}^0\mathfrak{X}^0(E[1]) \mid \delta(C^\infty(M)) \subseteq I_p C^\infty(M)\}$$

$$\begin{aligned}\mathfrak{g}^1_{LnA}(p,(1,l)) := \{\delta \in \mathfrak{X}^1(E[1]) \mid & \delta(C^\infty(M)) \subseteq I_p\Gamma(E_1^*[-1]),\\ & \delta^{(0)}(\Gamma(E_1^*[-1])) \subseteq I_p^l\Gamma(E_2^*[-2])\}\end{aligned}$$

$$\begin{aligned}\mathfrak{g}^2_{LnA}(p,(1,l)) := \{\delta \in \mathfrak{X}^2(E[1]) \mid & \delta^{(2)}(C^\infty(M)) \subseteq I_p\Gamma(S^2(E_1^*[-1])),\\ & \delta^{(1)}(C^\infty(M)) \subseteq I_p^{l+1}\Gamma(E_2^*[-2]),\\ & \delta^{(1)}(\Gamma(E_1^*[-1])) \subseteq I_p^l\Gamma(E_2^*[-2])\Gamma(E_1^*[-1])\},\end{aligned}$$

where the superscripts in parentheses correspond to the arity.

Definition 3.4.19. Let (E, Q) be a Lie n-algebroid over M. Let $l \geq 0$ be an integer. A point $p \in M$ is called a *fixed point of order (1,l)* if $Q \in \mathfrak{g}^1_{LnA}(p,(1,l))$.

Remark 3.4.20.

i) In terms of the anchor $\rho : E_1 \to TM$ and the multibrackets $\{\ell_k\}_{k\geq 1}$, a point $p \in M$ is a fixed point of order $(1,l)$ if $\rho_p = 0$, and if $\ell_1(\Gamma(E_2)) \subseteq I_p^l\Gamma(E_1)$.

ii) One may wonder why we include the order of vanishing of $\ell_1 : \Gamma(E_2) \to \Gamma(E_1)$ in the definition. In Section 3.4.5, we deal with singular foliations, where it will be essential that we can ensure stability of a fixed point of type $(1,l)$, rather than just a point where the anchor vanishes.

The following lemma shows that these subspaces actually yield a graded Lie subalgebra of $\mathfrak{g}$.

Lemma 3.4.21. *The subspaces defined above satisfy*

$$[\mathfrak{g}^0_{LnA}(p,(1,l)), \mathfrak{g}^i_{LnA}(p,(1,l))] \subseteq \mathfrak{g}^i_{LnA}(p,(1,l)),$$

$$[\mathfrak{g}^1_{LnA}(p,(1,l)), \mathfrak{g}^1_{LnA}(p,(1,l))] \subseteq \mathfrak{g}^2_{LnA}(p,(1,l))$$

for $i = 0, 1, 2$.

Similar to the Lie algebroid case, one can show that $\mathfrak{g}^0_{LnA}/\mathfrak{g}^0_{LnA}(p,(1,l)) \cong T_pM$, and $\mathfrak{g}^i_{LnA}/\mathfrak{g}^i_{LnA}(p,(1,l))$ for $i = 1, 2$ consists of jets at p of sections of some vector bundles. We will give a description of the complex below in remark 3.4.22.

iii) By the description above, the splittings can be chosen to be polynomial sections when restricting to a small neighborhood of $p \in M$.

iv) Fix a Lie n-algebroid structure $Q \in \mathfrak{g}_{LnA}(p,(1,l))$, i.e. a Lie n-algebroid structure on E such that $p \in M$ is a fixed point of order $(1,l)$ of Q.

Remark 3.4.22. We can give an explicit description of the complex $(\mathfrak{g}_{LnA}/\mathfrak{g}_{LnA}(p,(1,l)), \overline{[Q,-]})$ in degrees 0, 1 and 2 in terms of the multibrackets. After picking coordinates around the fixed point p, we may assume that $M = V$ is a vector space and p is the origin. Let $E = V \times \bigoplus_{i=1}^n \mathfrak{e}_i[i-1]$ a trivial bundle. Then the cochain spaces are isomorphic to

$$\begin{aligned}
\mathfrak{g}^0_{LnA}/\mathfrak{g}^0_{LnA}(p,(1,l)) &\cong V \\
\mathfrak{g}^1_{LnA}/\mathfrak{g}^1_{LnA}(p,(1,l)) &\cong \mathfrak{e}_1^* \otimes V \oplus J^{l-1}_p(E_2^* \otimes E_1) \\
\mathfrak{g}^2_{LnA}/\mathfrak{g}^2_{LnA}(p,(1,l)) &\cong S^2(\mathfrak{e}_1^*[-1]) \otimes V \\
&\qquad \oplus J^l_p(E_2^* \otimes TV) \oplus J^{l-1}_p(E_1^* \otimes E_2^* \otimes E_1).
\end{aligned}$$

For $v \in V$, the map

$$\overline{[Q,-]} : \mathfrak{g}^0_{LnA}/\mathfrak{g}^0_{LnA}(p,(1,l)) \to \mathfrak{g}^1_{LnA}/\mathfrak{g}^1_{LnA}(p,(1,l))$$

is defined by

$$\overline{[Q,-]}(v) = (d^\tau_{CE}(v), -v(\ell_1) + I^l_p\Gamma(E_2^* \otimes E_1)), \tag{3.13}$$

where we use that the bundles are trivialized. Here we recall that d^τ_{CE} is the Chevalley-Eilenberg differential associated to the linear holonomy representation $\tau : \mathfrak{e}_1 \to \mathrm{End}(T_pM)$ as in equation (3.1) and definition 3.2.5.
For $\alpha : \mathfrak{e}_1 \to V$ and $\delta : \Gamma(E_2) \to \Gamma(E_1)$, the map

$$\overline{[Q,-]} : \mathfrak{g}^1_{LnA}/\mathfrak{g}^1_{LnA}(p,(1,l)) \to \mathfrak{g}^2_{LnA}/\mathfrak{g}^2_{LnA}(p,(1,l))$$

is defined by

$$\begin{aligned}
\overline{[Q,-]}(\alpha, \delta + I^l_p\Gamma(E_2^* \otimes E_1)) = (d^\tau_{CE}(\alpha), \rho \circ \delta + \alpha \circ \ell_1 + I^{l+1}_p\Gamma(E_2^* \otimes TV), \\
\overline{[Q,-]}_{(2,1)}(\alpha, \delta + I^l_p)),
\end{aligned} \tag{3.14}$$

where the last term for $e \in \mathfrak{e}_1, f \in \mathfrak{e}_2$ is defined by

$$\overline{[Q,-]}_{(2,1)}(\alpha, \delta + I^l_p)(e,f) = \alpha(e)(\ell_1(f)) + \ell_2(e, \delta(f)) - \delta(\ell_2(e,f)) + I^l_p\Gamma(E_1). \tag{3.15}$$

The cocycles are therefore pairs $(\alpha, \delta + I_p^l\Gamma(E_2^* \otimes E_1))$, where α is a Chevalley-Eilenberg cocycle for $\mathfrak{e}_1$ with values in the representation V, and extending it to any section $\hat{\alpha}$, it satisfies

$$\rho \circ \delta + \hat{\alpha} \circ \ell_1 \in I_0^{l+1}\Gamma(E_2^* \otimes TV)$$

and for any $e \in \mathfrak{e}_1, f \in \mathfrak{e}_2$,

$$\alpha(e)(\ell_1(f)) + \ell_2(e, \delta(f)) - \delta(\ell_2(e, f)) \in I_p^l\Gamma(E_1).$$

We check that the assumptions 3.3.17a)-e) are satisfied.

a) By considerations similar to the ones for Lie algebroids, we choose the following topologies.
   - On $\mathfrak{g}_{LnA}^0$, we pick the C^∞-topology,
   - On $\mathfrak{g}_{LnA}^1$, we pick the $C^{\max\{1,l\}}$-topology,
   - On $\mathfrak{g}_{LnA}^2$, we pick the C^l-topology.

b) As $\partial = 0$, it is continuous.

c) Note that the continuity of the bracket is only needed in certain arity components, as the others disappear after taking the quotient. More precisely, the continuity of $[-,-] : \mathfrak{g}_{LnA}^1 \times \mathfrak{g}_{LnA}^1 \to \mathfrak{g}_{LnA}^2$ is only needed when restricting the domain to

$$(\pi_{E_2^*} \oplus \pi_{S^2(E_1^*[-1]})\mathfrak{g}_{LnA}^1 \pi_{E_1^*} \oplus \pi_{E_1^*}\mathfrak{g}_{LnA}^1 \pi_{C^\infty(M)},$$

and the codomain to

$$(\pi_{E_{-2}^*} \oplus \pi_{S^2(E_1^*[-1])})\mathfrak{g}_{LnA}^2 \pi_{C^\infty(M)} \oplus \pi_{E_1^* \otimes E_2^*}\mathfrak{g}_{LnA}^2 \pi_{E_{-1}^*},$$

where the π denote projections onto the respective component in $C^\infty(E[1])$. In these degrees, continuity is guaranteed by the choices above.

d) The gauge action of $\mathfrak{g}_{LnA}^0$ on $\mathfrak{g}_{LnA}^1$ is similar to item d) in Section 3.4.1.1. The degree 0 part is given by $\mathfrak{g}_{LnA}^0 = CDO\left(\bigoplus_{i=1}^n E_i^*\right)$, and these act by degree 0 automorphisms of the graded vector bundle E, covering the flow of the symbol of the differential operator. Hence the analogue of Lemma 3.3.12 still holds, and a neighborhood of the origin of $\mathfrak{g}_{LnA}^0/\mathfrak{g}_{LnA}^0(p, (1, l))$ corresponds to a neighborhood of $p \in M$.

e) Lemma 3.3.19 implies that the gauge action preserves Maurer-Cartan elements.

3.4.2.3 Applying the main theorem

Let $\mathfrak{g}_{LnA}$ and $\mathfrak{g}_{LnA}(p,(1,l))$ as above in equation (3.12) and Definition 3.4.18 respectively. Plugging this into the main theorem yields:

Theorem 3.4.23. *Let (E,Q) be a Lie n-algebroid over M. Let $p \in M$ be a fixed point of order $(1,l)$ for $l \geq 0$, that is, $Q \in \mathfrak{g}^1_{LnA}(p,(1,l))$. Assume that*

$$H^1(\mathfrak{g}_{LnA}/\mathfrak{g}_{LnA}(p,(1,l)), \overline{[Q,-]}) = 0.$$

Then for every open neighborhood U of $p \in M$, there exists a $C^{\max\{1,l\}}$-neighborhood $\mathcal{U}$ of Q such that for any Lie n-algebroid structure $Q' \in \mathcal{U}$ there is a family I in U of fixed points of order $(1,l)$ of Q' parametrized by an open neighborhood of

$$0 \in H^0(\mathfrak{g}_{LnA}/\mathfrak{g}_{LnA}(p,(1,l)), \overline{[Q,-]}).$$

Remark 3.4.24. This result gives a more refined criterium for stability of the fixed point p: instead of looking at E_1 with its almost Lie algebroid structure as in remark 3.4.17, we additionally take into account the map $\ell_1 : \Gamma(E_2) \to \Gamma(E_1)$. The relation between remark 3.4.17 and Theorem 3.4.23 is as follows: the projection of a vector field $X \in \mathfrak{X}^i(E[1])$ to the arity i component restricted to $C^\infty(E_1[1])$ defines a map

$$\mathrm{res} : \mathfrak{g}_{LnA} \to \mathfrak{X}(E_1[1]),$$

with $\mathrm{res}(\mathfrak{g}^j_{LnA}(p,(1,l)) \subseteq \mathfrak{X}^j_{p,1}(E_1[1])$ for $j = 0,1,2$. It therefore induces a map

$$\overline{\mathrm{res}} : \mathfrak{g}^j_{LnA}/\mathfrak{g}^j_{LnA}(p,(1,l)) \to \mathfrak{X}^j(E_1[1])/\mathfrak{X}^j_{p,1}(E_1[1]).$$

Now given a Lie n-algebroid structure $Q \in \mathfrak{g}^1_{LnA}(p,(1,l))$ with corresponding almost Lie algebroid structure $Q^{(1)} \in \mathfrak{X}^1_{p,1}(E_1[1])$, the map $\overline{res}$ is compatible with the differentials $\overline{[Q,-]}$ and $\overline{[Q^{(1)},-]}$ of $\mathfrak{g}_{LnA}/\mathfrak{g}_{LnA}(p,(1,l))$ and $\mathfrak{X}(E_1[1])$ respectively in degrees 0 and 1. Consequently, $\overline{res}$ descends to cohomology:

$$H^1(\overline{\mathrm{res}}) : H^1(\mathfrak{g}_{LnA}/\mathfrak{g}_{LnA}(p,(1,l)), \overline{[Q,-]}) \to H^1(\mathfrak{X}(E_1[1])/\mathfrak{X}_{p,1}(E_1[1]), \overline{[Q^{(1)},-]}).$$

For $l = 0$, $H^1(\overline{\mathrm{res}})$ is injective, reflecting the fact that any Lie n-algebroid (E,Q) has an underlying almost Lie algebroid structure $(E_1, Q^{(1)})$.
For $l > 0$, the map is no longer injective, but vanishing of

$$H^1(\mathfrak{g}_{LnA}/\mathfrak{g}_{LnA}(p,(1,l)), \overline{[Q,-]})$$

also guarantees the existence of fixed points of order $(1,l)$, which the vanishing of

$$H^1(\mathfrak{X}(E_1[1])/\mathfrak{X}_{p,1}(E_1[1]), \overline{[Q^{(1)},-]})$$

does not.

3.4.3 Higher order fixed points of Lie n-algebroids

As for Lie algebroids, there are several examples of Lie n-algebroids, for which Theorem 3.4.23 fails in a trivial way because the anchor of the Lie n-algebroid vanishes up to higher order, causing components of Q to vanish to first order. For this reason, we extend Theorem 3.4.23 to so called fixed points of order (k, l), where k and l are integers.

3.4.3.1 The ingredients

We show that we are in the setting of assumptions 3.3.17.

i) As we still work with Lie n-algebroid structures on a graded vector bundle $E = \bigoplus_{i=1}^{n} E_i[i-1]$, take the graded Lie algebra $\mathfrak{g}_{LnA}$ as in equation (3.12).

ii) We define the graded Lie subalgebra, corresponding to $p \in M$ being a fixed point of order (k, l).

Let $p \in M$, $k \geq 0, 0 \leq l \leq 2k-2$ integers.

Definition 3.4.25. Define

$$\mathfrak{g}^0_{LnA}(p,(k,l)) := \{\delta \in {}^0\mathfrak{X}^0(E[1]) \mid \delta(C^\infty(M)) \subseteq I_p\}$$

$$\begin{aligned}
\mathfrak{g}^1_{LnA}(p,(k,l)) := \{\delta \in \mathfrak{X}^1(E[1]) \mid \, & \delta(C^\infty(M)) \subseteq I_p^k \Gamma(E_1^*[-1]), \\
& \delta^{(1)}(\Gamma(E_1^*[-1])) \subseteq I_p^{k-1}\Gamma(S^2(E_1^*[-1])), \\
& \delta^{(0)}(\Gamma(E_1^*[-1])) \subseteq I_p^{l}\Gamma(E_2^*[-2]), \\
& \delta^{(2)}(\Gamma(E_2^*[-2])) \subseteq I_p^{2k-2-l}\Gamma(S^3(E_1^*[-1]))\}
\end{aligned}$$

$$\begin{aligned}
\mathfrak{g}^2_{LnA}(p,(k,l)) := \{\delta \in \mathfrak{X}^2(E[1]) \mid \, & \delta^{(2)}(C^\infty(M)) \subseteq I_p^{2k-1}\Gamma(S^2(E_1^*[-1])), \\
& \delta^{(1)}(C^\infty(M)) \subseteq I_p^{k+l}\Gamma(E_2^*[-2]), \\
& \delta^{(2)}(\Gamma(E_1^*[-1])) \subseteq I_p^{2k-2}\Gamma(S^3(E_1^*[-1]))\}.
\end{aligned}$$

Then:

Definition 3.4.26. Let (E, Q) be a Lie n-algebroid over M. We say that $p \in M$ is a *fixed point of order* (k, l) if $Q \in \mathfrak{g}^1_{LnA}(p,(k,l))$.

Remark 3.4.27. Relating to remark 3.4.15iv), these conditions in classical terms are equivalent to the following. For $x, y, z \in \Gamma(E_1)$, $e \in \Gamma(E_2)$, we require the following:

$$\rho(x) \in I_p^k \mathfrak{X}(M),$$

$$\ell_2(x, y) \in I_p^{k-1} \Gamma(E_1),$$

$$\ell_1(e) \in I_p^l \Gamma(E_1),$$

$$\ell_3(x, y, z) \in I_p^{2k-l-2} \Gamma(E_2).$$

The first condition is what we are interested in, while the second condition is necessary to make things work for the same reason as for Lie algebroids. The seemingly odd last condition can be motivated by the equation

$$Q^2(\alpha) = 0, \tag{3.16}$$

for $\alpha \in \Gamma(E_1^*)$: when looking at the weight 2 part of equation (3.16), it is equivalent to

$$\ell_2(x, \ell_2(y, z)) + \text{cycl.}(x, y, z) = \ell_1(\ell_3(x, y, z)).$$

As the left hand side of the equation lies in $I_p^{2k-2}\Gamma(E_1)$, one way to ensure the same holds for the right hand side is to require that $\ell_3(x, y, z) \in I_p^{2k-l-2}\Gamma(E_2)$.

Lemma 3.4.28. *The subspaces defined above satisfy*

$$[\mathfrak{g}^0_{LnA}(p, (k, l)), \mathfrak{g}^i_{LnA}(p, (k, l))] \subseteq \mathfrak{g}^i_{LnA}(p, (k, l)),$$

$$[\mathfrak{g}^1_{LnA}(p, (k, l)), \mathfrak{g}^1_{LnA}(p, (k, l))] \subseteq \mathfrak{g}^2_{LnA}(p, (k, l))$$

for $i = 0, 1, 2$.

This tells us that $\mathfrak{g}_{LnA}(p, (k, l))$ is indeed a graded Lie subalgebra (if as before, we set $\mathfrak{g}^{\geq 3}_{LnA}(p, (k, l)) = \mathfrak{g}^{\geq 3}_{LnA}$), whose Maurer-Cartan elements are precisely those Lie algebroid structures for which $p \in M$ is a fixed point of order (k, l).

Remark 3.4.29. After picking a suitably small coordinate neighborhood of $p \in M$ such that E is a trivial bundle, we can identify the cochain spaces of the complex $\mathfrak{g}_{LnA}/\mathfrak{g}_{LnA}(p, k)$ in the relevant degrees with the following:

$$\mathfrak{g}^0_{LnA}/\mathfrak{g}^0_{LnA}(p,(k,l)) \cong T_pM$$

$$\mathfrak{g}^1_{LnA}/\mathfrak{g}^1_{LnA}(p,(k,l)) \cong J^{k-1}_p(E^*_1[-1]\otimes TM)\oplus J^{k-2}_p(S^2(E^*_1[-1])\otimes E_1[1])$$
$$\oplus J^{l-1}_p(E^*_2[-2]\otimes E_1[1])\oplus J^{2k-l-3}_p(S^3(E^*_1[-1])\otimes E_2[2])$$

$$\mathfrak{g}^2_{LnA}/\mathfrak{g}^2_{LnA}(p,(k,l)) \cong J^{2k-2}(S^2(E^*_1[-1])\otimes TM)$$
$$\oplus J^{2k-3}_p(S^3(E^*_1[-1])\otimes E_1[1])\oplus J^{k+l-1}_p(E^*_2[-2]\otimes TM).$$

iii) It follows that by restricting to a sufficiently small coordinate neighborhood of $p\in M$ over which E trivializes as a graded vector bundle, we can choose splittings

$$\sigma_i : \mathfrak{g}^i_{LnA}/\mathfrak{g}^i_{LnA}(p,(k,l)) \to \mathfrak{g}^i_{LnA}$$

for $i=0,1$.

iv) Now fix a Lie $n-$algebroid structure $Q\in\mathfrak{g}^1_{LnA}(p,(k,l))$.

We check that the data satisfies assumptions 3.3.17a)-d).

a) We pick the following topologies.
  - On $\mathfrak{g}^0_{LnA}$, we pick the C^∞-topology,
  - On $\mathfrak{g}^1_{LnA}$, we pick the $C^{\max\{2k-1,k+l-1\}}$-topology,
  - On $\mathfrak{g}^2_{LnA}$, we pick the $C^{\max\{2k-2,k+l-1\}}$-topology.

b) As $\partial = 0$, it is continuous.

c) As for fixed points of order $(1,l)$ it is only certain components of $\mathfrak{g}^1_{LnA}$ and $\mathfrak{g}^2_{LnA}$ we need continuity of the bracket in. More precisely, for the domain, we may restrict to

$$(\pi_{E^*_2}\oplus\pi_{S^2(E^*_1[-1]})\mathfrak{g}^1_{LnA}\pi_{E^*_1}\oplus\pi_{E^*_1}\mathfrak{g}^1_{LnA}\pi_{C^\infty(M)}\oplus\pi_{S^3(E^*_1[-1])}\mathfrak{g}^1_{LnA}\pi_{E^*_2[-2]},$$

while for the codomain, we need

$$(\pi_{S^2(E^*_1[-1])}\oplus\pi_{E^*_2})\mathfrak{g}^2_{LnA}\pi_{C^\infty(M)}\oplus\pi_{S^3(E^*_1[-1])}\mathfrak{g}^2_{LnA}\pi_{E^*_1}.$$

The choices in a) guarantee the continuity.

d) The gauge action is the same as for fixed points of order $(1,l)$. In particular, a neighborhood of 0 in $\mathfrak{g}_{LnA}/\mathfrak{g}_{LnA}(p,(k,l))$ corresponds to a neighborhood of $p\in M$.

e) Lemma 3.3.19 implies that the gauge action preserves Maurer-Cartan elements.

3.4.3.2 Applying the main theorem

Let $\mathfrak{g}_{LnA}$ and $\mathfrak{g}_{LnA}(p,(k,l))$ be as in equation (3.12) and Definition 3.4.25 respectively. Applying Theorem 3.3.20, we obtain:

Theorem 3.4.30. *Let (E,Q) be a Lie n-algebroid over M. Let $p \in M$ be a fixed point of order (k,l) for $k \geq 0$, $0 \leq l \leq 2k-2$, that is, $Q \in \mathfrak{g}^1_{LnA}(p,(k,l))$. Assume that*

$$H^1(\mathfrak{g}_{LnA}/\mathfrak{g}_{LnA}(p,(k,l)), \overline{[Q,-]}) = 0.$$

Then for every open neighborhood U of $p \in M$, there exists a $C^{\max\{2k-1,k+l-1\}}$-neighborhood $\mathcal{U}$ of $Q \in \mathfrak{g}_{LnA}$ such that for any Lie n-algebroid structure $Q' \in \mathcal{U}$ there is a family $I \subseteq U$ of fixed points of order (k,l) of Q' parametrized by an open neighborhood of

$$0 \in H^0(\mathfrak{g}_{LnA}/\mathfrak{g}_{LnA}(p,(k,l)), \overline{[Q,-]})$$

.

3.4.4 Examples

In this section we compute the various cohomologies we encountered in some explicit examples.

Example 3.4.31. Let $M = \mathbb{R}^2$. Consider the Lie algebroid

$$\mathfrak{gl}_2 \times M,$$

associated to the standard action on M. The origin p is a fixed point of order 1 of the Lie algebroid with singular foliation, and we will show that it is stable for nearby Lie algebroid structures using Theorem 3.2.8. The relevant cohomology in this case is given by

$$H^1_{CE}(\mathfrak{gl}_2, T_pM).$$

This cohomology vanishes as for any $\alpha : \mathfrak{gl}_2 \to T_pM$, the cocycle condition reads

$$\alpha([x,y]) = x\alpha(y) - y\alpha(x)$$

for every $x, y \in \mathfrak{gl}_2$. Plugging in $x = \mathrm{id}$, we see that $\alpha = d^{\tau}_{CE}(\alpha(\mathrm{id}))$.
Therefore, using Theorem 3.2.8, the origin is a stable fixed point for nearby Lie algebroid structures.
But we can say more. There is a natural Lie 2-algebroid (E,Q) associated to this action [LGLS20, Example 3.33, Example 3.98], such that the origin is a fixed point of order (1,1). We will apply Theorem 3.4.23 to show that the fixed

point is also stable for Lie 2-algebroid structures close to this one. The only information we need is that E_1 is the trivial bundle with fiber $\mathfrak{gl}_2$, such that the binary bracket agrees with the Lie bracket of the action Lie algebroid, E_2 is a trivial bundle with fiber $\mathbb{R}^2$ and that the complex

$$0 \longrightarrow \Gamma(E_2) \xrightarrow{\ell_1} \Gamma(E_1) \xrightarrow{\rho} \rho(\Gamma(E_1)) \longrightarrow 0$$

is exact.
Indeed, following remark 3.4.22, we can compute the cohomology relevant to Theorem 3.4.23. If we take a cocycle $(\alpha, \delta + I_p\Gamma(E_2^* \otimes E_1)) \in \mathfrak{X}^1(E[1])/\mathfrak{X}^1_{p,(1,1)}(E[1])$, then the vanishing of $H^1_{CE}(\mathfrak{gl}_2, T_pM)$ implies that $\alpha = d^{\tau}_{CE}(v)$ for some $v \in T_pM$, or more explicitly, we have $\alpha = -v(\rho)(p) \in (E_1)^*_p \otimes T_pM$.
The second cocycle condition of in equation (3.14) now reads that

$$-v(\rho)(\ell_1(e)) + \rho(\delta(e)) \in I_p^2\mathfrak{X}(M)$$

for every section $e \in \Gamma(E_2)$. If we now take e to be a constant section, then we know in particular that

$$-v(\rho)(\ell_1(e)) - \rho(v(\ell_1(e)) = -v(\rho(\ell_1(e))) = 0.$$

So we find

$$\rho(\delta(e) + v(\ell_1)(e)) \in I_p^2\mathfrak{X}(M).$$

As this means that the first order Taylor expansion of this expression vanishes, we look at the linear part of this:

$$0 = (\rho(\delta(e) + v(\ell_1)(e)))^{(1)} = \rho^{(1)}(\delta(e)(p) + v(\ell_1)(e)(p)).$$

However, note that ρ has linear coefficient functions. This means that $\rho^{(1)} = \rho$, and it follows that the constant section $\delta(e)(p) + v(\ell_1(e))(p) \in \ker(\rho : \Gamma(E_1) \to \mathfrak{X}(M))$. However, as $\ker(\rho) = \mathrm{im}(\ell_1)$ at the level of sections, and all sections in the image of ℓ_1 vanish at the origin, it follows that $\delta(p) + v(\ell_1)(p) = 0$, which means that $\delta = -v(\ell_1) + I_p\Gamma(E_2^* \otimes E_1)$, showing that $H^1(\mathfrak{g}_{LnA}/\mathfrak{g}_{LnA}(p,(1,1)), \overline{[Q,-]})$ vanishes (without needing the third cocycle condition!), and that p is stable as a fixed point of order $(1,1)$ of the Lie 2-algebroid by Theorem 3.4.23.

A similar example is given by the special linear subalgebra.

Example 3.4.32. Let $M = \mathbb{R}^2$. Consider the Lie algebroid

$$\mathfrak{sl}_2 \times M$$

associated to the standard action on M. The origin p is a fixed point of order 1 of the Lie algebroid and we will show that it is stable using Theorem 3.2.8. The relevant cohomology in this case is given by

$$H^1_{CE}(\mathfrak{sl}_2, T_pM),$$

which vanishes by Whitehead's first lemma, as $\mathfrak{sl}_2$ is simple. Therefore, the origin is a stable fixed point for nearby Lie algebroid structures by Theorem 3.2.8.
Again, there is a natural Lie 2-algebroid (E, Q) associated to this action [LGLS20, Example 3.30, Example 3.96], such that the origin p is a fixed point of order $(1, 2)$, and it can be shown that $H^1(\mathfrak{g}_{LnA}/\mathfrak{g}_{LnA}(p,(1,2)), \overline{[Q,-]})$ vanishes in the same way as in example 3.4.31. Consequently, p is stable as a fixed point of order $(1, 2)$ of this Lie 2-algebroid by Theorem 3.4.23.

Example 3.4.33. Let $M = \mathbb{R}^3$, and consider the function $\phi \in C^\infty(M)$ given by

$$\phi(x, y, z) = \frac{1}{6}(x^3 + y^3 + z^3).$$

There is a Lie 2-algebroid, for which the image of the anchor map is given by all vector fields preserving ϕ, as in [LGLS20, Example 3.101]. The origin p is a fixed point of order $(2, 2)$, but p is *not* stable! The Lie 2-algebroid is given by the trivial rank 3 bundle in degree -1 with frame $\{e_1, e_2, e_3\}$, and the trivial line bundle in degree -2 with frame $\{f_1\}$ (with corresponding dual frames $\{e^1, e^2, e^3\}$ and $\{f^1\}$ respectively). If Q denotes the homological vector field of this Lie 2-algebroid, then it can be shown that

$$Q_\epsilon = Q + (e^1 - e^2) \otimes \epsilon\partial_x - f^1 \otimes \epsilon e_3$$

is still a Lie n-algebroid structure, but has no zero-dimensional leaves for $\epsilon < 0$, where $f^1 \otimes \epsilon e_3$ is viewed as an arity 0 degree 1 vector field. By Theorem 3.4.30, we see that the relevant cohomology $H^1(\mathfrak{g}_{LnA}/\mathfrak{g}_{LnA}(p,(2,2)), \overline{[Q,-]})$ is nonzero. Indeed, one can check that $\frac{1}{\epsilon}(Q_\epsilon - Q)$ is a nontrivial cocycle.

3.4.5 Fixed points of singular foliations

In this section we apply the results of the previous section to singular foliations. We work towards a cohomological stability criterium for certain singular foliations (Proposition 3.4.42). We then apply this to obtain a formal rigidity criterium for foliations induced by a linear representation of a semisimple Lie algebra (Corollary 3.4.46). We start with a general lemma, which extends examples 3.4.31 and 3.4.32, giving a sufficient condition for when the cohomological assumption of Theorem 3.4.23 is satisfied.

Lemma 3.4.34. *Let V be a finite-dimensional vector space, and let $(E = V \times \bigoplus_{i=1}^{n} \mathfrak{g}_i[i-1], Q)$ be a Lie n-algebroid over V, such that the origin p is a fixed point of order $(1,l)$. Assume that*

- *The coefficient functions of the anchor $\rho_Q : \Gamma(E_1) \to \mathfrak{X}(V)$ are linear,*
- $H^1_{CE}(\mathfrak{g}_1, V) = 0$,
- $\ker(\rho_Q) = im(\ell_1^Q)$.

Then

$$H^1(\mathfrak{g}_{LnA}/\mathfrak{g}_{LnA}(p,(1,l)), \overline{[Q,-]}) = 0.$$

Proof. We use the description of $H^1(\mathfrak{g}_{LnA}/\mathfrak{g}_{LnA}(p,(1,l)), \overline{[Q,-]})$ as in remark 3.4.22.
The main ingredients of the proof are already present in example 3.4.31. Given a cocycle $(\alpha, \delta + I_p^l\Gamma(E_2^* \otimes E_1)) \in \mathfrak{g}^1_{LnA}/\mathfrak{g}^1_{LnA}(p,(1,l))$, we eventually find that

$$\rho(\delta(e) + v(\ell_1(e))) \in I_p^{l+1}\mathfrak{X}(V)$$

for every constant section $e \in \Gamma(E_2)$. Taking the s-homogeneous part of this expression for $s = 1, \dots, l-1$, we see that it is equal to

$$0 = \rho^{(1)}(\delta^{(s-1)}(e)),$$

and for the l-homogeneous part, we find

$$0 = \rho^{(1)}(\delta^{(l-1)}(e) + v(\ell_1^{(l)})(e)).$$

As $\rho^{(1)} = \rho$, and $\ker(\rho) = \mathrm{im}(\ell_1) \subseteq I_p^l\Gamma(E_1)$, it follows that the s-homogeneous part of δ for $s = 0, \dots, l-2$ vanishes, while the $l-1$-homogenous part is given by $-v(\ell_1^{(l)})$. As $v(\ell_1) \in I_p^{l-1}\Gamma(E_2^* \otimes E_1)$, we see that $\delta = -v(\ell_1) + I_p^l\Gamma(E_2^* \otimes E_1)$, which concludes the proof. $\square$

Remark 3.4.35. Note that the linearity of ρ only comes into play when we want to use properties of the kernel of ρ. In fact a weaker condition which can replace linearity of ρ and $\ker(\rho) = \mathrm{im}(\ell_1)$ is that the linear part $\rho^{(1)}$ of ρ, which can be viewed as a map between sections of E_1 and vector fields on V has kernel contained in $I_p^l\Gamma(E_1)$.

Recall the following definition.

Definition 3.4.36. Let M be a smooth manifold. A *singular foliation* on M is a subsheaf $\mathcal{F} \subseteq \mathfrak{X}_M$ which is locally finitely generated, and involutive. Denote a foliated manifold by $(M, \mathcal{F})$.
We say that $p \in M$ is a fixed point of $\mathcal{F}$ if $\mathcal{F} \subseteq I_p\mathfrak{X}_M$.

Just as for Lie algebroids, associated to a fixed point $p \in M$ of a singular foliation $\mathcal{F}$, there is a Lie algebra $\mathfrak{g}_p$ and a representation of $\mathfrak{g}_p$ on T_pM:

Lemma 3.4.37 ([AS09],[AZ12])**.** *Let $(M, \mathcal{F})$ be a foliated manifold and $p \in M$ a fixed point of $\mathcal{F}$.*

i) Let $\mathcal{F}_p$ denote the stalk of $\mathcal{F}$ at p. Then the induced Lie bracket on $\mathcal{F}_p$ descends to the finite-dimensional vector space

$$\mathfrak{g}_p := \mathcal{F}_p / I_p \mathcal{F}_p.$$

ii) The map

$$\tau : \mathfrak{g}_p \to \mathrm{End}(T_pM)$$

given by

$$\tau(x)(v) = [\tilde{x}, \tilde{v}](p)$$

is a well-defined representation of $\mathfrak{g}_p$ called the linear holonomy representation, *where $x \in \mathfrak{g}_p, v \in T_pM$, and $\tilde{x}, \tilde{v}$ are extensions to elements of $\mathcal{F}$ and $\mathfrak{X}(M)$ respectively.*

Remark 3.4.38. Note that a Lie n-algebroid (E, Q) induces a singular foliation on the base manifold M, which is the image of the anchor map at the level of sections.

The following definition was proposed by Camille Laurent-Gengoux and Sylvain Lavau [LGL].

Definition 3.4.39. Let $(M, \mathcal{F})$ be a foliated manifold. A *isomodule deformation* of $\mathcal{F}$ is a singular foliation $\mathcal{F}_U$ on $M \times U$, where $0 \in U \subseteq \mathbb{R}^n$ is an open subset, such that

a) For every $p \in U$, $\mathcal{F}^p$ is tangent to $M \times \{p\}$, where $\mathcal{F}^p$ is the restriction of $\mathcal{F}_U$ to $M \times \{p\}$,

b) $\mathcal{F}^0 = \mathcal{F}$,

together with an isomorphism of $C^\infty_{M\times U}$-modules $\phi : C^\infty_{M\times U} \otimes_{C^\infty_M} \mathcal{F} \to \mathcal{F}_U$.

This allows us to define a notion of stability of fixed points of a singular foliation.

Definition 3.4.40. Let $(M, \mathcal{F})$ be a foliated manifold, and let $p \in M$ be a zero-dimensional leaf of $\mathcal{F}$. We say that p is *stable* if for every isomodule deformation $(M \times U, \mathcal{F}_U)$, for every neighborhood $p \in V \subseteq M$, there is a neighborhood $0 \in W \subseteq U$, such that $\mathcal{F}^q$ has a zero-dimensional leaf in V for $q \in W$.

The following lemma was proven by Camille Laurent-Gengoux and Sylvain Lavau [LGL]. We formulate a weaker version and include the proof in the appendix for completeness.

Lemma 3.4.41. *Let $(V, \mathcal{F})$ be a vector space equipped with a* linear *foliation, and $(V \times U, \mathcal{F}_U)$ an isomodule (not necessarily linear) deformation. Then there exists a geometric resolution (see [LGLS20]) of $\mathcal{F}$*

$$0 \longrightarrow \Gamma(E_n) \longrightarrow \dots \longrightarrow \Gamma(E_1) \longrightarrow \mathcal{F} \longrightarrow 0, \tag{3.17}$$

such that the differential has polynomial coefficient functions. Moreover, there exists a Lie n-algebroid structure Q on $\oplus_{i=1}^n p^* E_i$, where $p : V \times U \to V$ is the projection, with the following properties:

- *The unary bracket extends the complex* (3.17)*,*
- *Q induces the foliation $\mathcal{F}_U$,*
- *The sequence of $C^\infty_{M\times U}$-modules*

  $$0 \longrightarrow \Gamma(p^*E_n) \longrightarrow \dots \longrightarrow \Gamma(p^*E_1) \longrightarrow \mathcal{F}_U \longrightarrow 0,$$

  is exact,
- *The restriction of Q to $V \times \{0\}$ is a Lie n-algebroid structure on E inducing $\mathcal{F}$.*

Using the results of this section, we obtain a stability result for linear foliations.

Proposition 3.4.42. *Let $(V, \mathcal{F})$ be a vector space, equipped with a linear foliation. Assume that*

$$H^1_{CE}(\mathfrak{g}, V) = 0,$$

where $\mathfrak{g}$ is the isotropy Lie algebra of $\mathcal{F}$ at 0 acting by the linear holonomy representation. Then the origin is a stable fixed point for all isomodule deformations.

Proof. Let $(V \times U, \mathcal{F}_U)$ be a isomodule deformation of $\mathcal{F}$, and pick a Lie n-algebroid (p^*E, Q) inducing $\mathcal{F}_U$ as in Lemma 3.4.41. By [LGLS20, Theorem 2.3.5], we may assume that the differential on the complex vanishes in 0 up to finite order l. In this case, the fiber $(E_1)_0$ over $0 \in V$ has the same dimension as the isotropy Lie algebra by [LGLS20, Proposition 4.14].
Denote by $\rho : \Gamma(E_1) \to \mathfrak{X}(V)$ the map inducing the foliation.

Claim. We may assume that ρ has linear coefficient functions.

Proof of claim. To see this, pick linear generators $\{X_i\}_{i=1}^r$ for $\mathcal{F}$ which are linearly independent over $\mathbb{R}$, and consider their images $\{e_i\}_{i=1}^r$ in $\mathfrak{g}$. It is clear that the e_i generate $\mathfrak{g}$. However, they are also linearly independent: if $a_i \in \mathbb{R}$ are such that

$$\sum_{i=1}^r a_i e_i = 0 \in \mathfrak{g},$$

then

$$\sum_{i=1}^r a_i X_i \in I_0\mathcal{F} \subseteq I_0^2\mathfrak{X}(M).$$

As the X_i are linear and were assumed to be linearly independent over $\mathbb{R}$, it follows that the a_i must be zero.
Pick preimages s_i for X_i under ρ. The diagram

$$\begin{array}{ccc} \Gamma(E_1) & \xrightarrow{\rho} & \mathcal{F} \\ \downarrow & & \downarrow \\ \Gamma(E_1)/I_0\Gamma(E_1) & \xrightarrow{\overline{\rho}} & \mathcal{F}/I_0\mathcal{F} \end{array}$$

commutes, and the bottom row is a map from $(E_1)_0$ to $\mathfrak{g}$, sending $s_i(0)$ to e_i. This shows that s_i form a local frame around 0, and $\rho(s_i) = X_i$ is a linear vector field, which proves the claim. □

By the choice of Lie n-algebroid (p^*E, Q) of $\mathcal{F}_U$, the restriction (E, Q_0) to $V \times \{0\}$ has a fixed point of order $(1, l)$, and satisfies the assumptions of Lemma 3.4.34. So given a neighborhood $W \subseteq V$ of the origin, there exists a neighborhood $\mathcal{U}$ of Q_0 in the space of Lie n-algebroid structures on E such that all $Q' \in \mathcal{U}$ have a fixed point $q \in W$.
Finally, as the map

$$U \to \{\text{Lie } n\text{-algebroid structures on } E\}$$

given by

$$p \mapsto Q_p$$

is continuous, the result follows. □

In particular, we have:

Corollary 3.4.43. *Let $(V, \mathcal{F})$ be a vector space equipped with a foliation that has linear generators, such that all vector fields in $\mathcal{F}$ vanish at the origin. If the isotropy Lie algebra $\mathfrak{g}$ of $\mathcal{F}$ in $0 \in V$ is semisimple, then the origin is a stable fixed point for all isomodule deformations.*

Proof. For semisimple Lie algebras $\mathfrak{g}$, $H^1_{CE}(\mathfrak{g}, -)$ is identically zero for finite-dimensional representations by Whitehead's first lemma. □

Remark 3.4.44.

- Under the assumption that the foliation $\mathcal{F}$ we start with is linear, we obtained a simplified criterium for stability of the origin, depending only on the foliation, and not on the chosen Lie n-algebroid inducing it.
- If $\mathcal{F}$ is arbitrary, with a fixed point, and $\mathcal{F}_U$ is a deformation which admits a geometric resolution as in [LGLS20, Definition 2.1], hence a Lie n-algebroid inducing $\mathcal{F}_U$, then restricting to $\mathcal{F}_0$ one gets a Lie n-algebroid inducing $\mathcal{F}$. Now Theorem 3.4.23 can be applied to the latter Lie n-algebroid if $H^1(\mathfrak{g}_{LnA}/\mathfrak{g}_{LnA}(p,(1,l)), \overline{[Q,-]}) = 0$.

If $(M, \mathcal{F})$ is a foliated manifold, and $p \in M$ a fixed point stable under module-stable deformations, let $(M \times U, \mathcal{F}_U)$ be such a module-stable deformation. In particular, for $W \subseteq U$ as in Proposition 3.4.42, $q \in W$ implies that $\mathcal{F}_q$ has a fixed point p' in M. This gives rise to two questions.

- Can we describe the isotropy Lie algebra of $\mathcal{F}_q$ in p'?
- If so, what can we say about the linear holonomy representation?

The following proposition provides an answer for both of these questions when $\mathfrak{g}$ is semisimple.

Proposition 3.4.45. *Let $(V, \mathcal{F})$ be a vector space equipped with a foliation that has linear generators, such that the isotropy Lie algebra is semisimple, so that the origin $0 = p \in V$ is a stable fixed point.*
Let $(V \times U, \mathcal{F}_U)$ be a isomodule deformation, such that for any $q \in U$, the foliation $\mathcal{F}_q$ has a fixed point $p'(q) \in V$.
Then for a possibly smaller neighborhood $W \subseteq U$, the isotropy Lie algebra at $p'(q)$ is isomorphic to $\mathfrak{g}$. Moreover, the linear holonomy representations are isomorphic.

Proof. As $\mathcal{F}_U$ is a isomodule deformation, let (p^*E, Q) be a Lie n-algebroid inducing $\mathcal{F}_U$ as in Lemma 3.4.41, for which the unary bracket vanishes at $(p, 0) \in V \times U$ up to order $l \in \mathbb{Z}_{>0}$.
As $l > 0$, by [LGLS20, Proposition 4.14], $((E_1)_{p'}, (\ell_2)_{(p',q)})$ is exactly the isotropy Lie algebra of $\mathcal{F}^q$ in p', where ℓ_2 is the binary bracket of (p^*E, Q). As we may assume E_1 is a trivial bundle we denote $(E_1)_p$ by $\mathfrak{h}$, and we consider the map

$$V \times U \to \wedge^2 \mathfrak{h}^* \otimes \mathfrak{h}$$

given by
$$(v,q) \mapsto (\ell_2)_{(v,q)},$$
where the right hand side should be seen as the map which extends $x_1, x_2 \in \mathfrak{h}$ to a constant section, applies the binary bracket of the Lie n-algebroid, and then evaluates it on $(v,q) \in V \times U$. As semisimple Lie algebras are rigid by [NR67b, Theorem 7.1] and Whitehead's second lemma, there is a neighborhood O of $(\ell_2)_{(p,0)}$, such that every Lie algebra structure in O is isomorphic to $(\ell_2)_{(p,0)}$. Hence there is a neighborhood D of $(p,0)$ such that for every pair $(p',q) \in D$ such that p' is a fixed point of $\mathcal{F}^q$, its isotropy Lie algebra is isomorphic to $\mathfrak{g}$. For the assertion about the linear holonomy representation the proof is analogous: we note that the linear holonomy representation is the linearization of the anchor, and that representations of semisimple Lie algebras are rigid by [NR67a, Theorem A] and Whitehead's first lemma. □

Proposition 3.4.45 says something about the first order approximation. If $\mathcal{F}_U$ is a deformation such that for every $q \in U$, $\mathcal{F}^q$ is linearizable, this yields a rigidity result for such a deformation.
There is a sufficient condition for a foliation with semisimple isotropy to be formally linearizable around a fixed point.

Corollary 3.4.46. *Let $(V, \mathcal{F})$ be a vector space equipped with a foliation with linear generators, such that the isotropy Lie algebra $\mathfrak{g}$ is semisimple. Let $(V \times U, \mathcal{F}_U)$ be a module-stable deformation such that $\mathcal{F}^q$ has analytic generators for every $q \in U$. For some neighborhood $W \subseteq U$ of the origin, for every $q \in W$, there is a formal diffeomorphism of $\phi_q : V \to V$ such that $\phi_q^* \mathcal{F}^q = \mathcal{F}$.*

Proof. It suffices to show that a foliation with analytic generators with semisimple isotropy in a fixed point is formally linearizable. This follows from [LGR21, Corollary 2.24]: to apply this result, we need to show that the linear holonomy representation $\mathfrak{g} \to \mathrm{End}(V)$ is faithful. The kernel is an ideal, hence it is itself a semisimple Lie algebra. However, [LGR21, Theorem 1.10] implies that the kernel is nilpotent. As any nilpotent and semisimple Lie algebra is trivial, this shows that the linear holonomy representation is faithful. □

3.5 Stability under additional structure

In the previous sections we have considered Lie n-algebroids, and applied Theorem 3.3.20 to give a sufficient condition for when a fixed point of some type of a Lie n-algebroid is stable.
Now suppose we are given a Lie algebroid with some additional structure. A

natural question we can ask is if we can refine the criterium when we only look at Lie algebroid structures which also have this additional structure. For instance, given a Lie algebroid structure on T^*M, we can require the Lie algebroid differential on $\mathfrak{X}^\bullet(M)$ to be a derivation of the Schouten-Nijenhuis bracket. Is there a theorem similar to Theorem 3.4.11 when we only allow Lie algebroid structures on T^*M which are in addition derivations of the Schouten bracket? Of course, it is known that such Lie algebroid structures are precisely the Poisson structures on M, so [CF10] and [DW06] contain results on it.
In the first subsection, we will address the question above by fixing a Lie algebroid (A, d_A), and apply Theorem 3.3.20 to give a stability criterium for fixed points of a Lie algebroid structure d_{A^*} on A^* such that $((A, d_A), (A^*, d_{A^*}))$ is a Lie bialgebroid. We then apply this result to obtain the following:

- First, by taking $A = TM$ with its standard Lie algebroid structure $d_A = d_{dR}$, recover the result from [DW06] for Poisson manifolds (Corollary 3.5.10).
- When $Z \subseteq M$ is a hypersurface, we let $A = {}^bTM$ be the b-tangent bundle. In this case we obtain a result for zeros of self-commuting b-bivector fields (Theorems 3.5.12 and 3.5.16).
- When $N : TM \to TM$ is a Nijenhuis tensor, we obtain a result for Lie algebroid structures near a Poisson structure compatible with N (Theorem 3.5.20), which we then refine to a result dealing only with Poisson-Nijenhuis structures (Theorem 3.5.21). As application of this, we obtain a stability result for fixed points of holomorphic Poisson structures.

In the second subsection, we look at Courant algebroids and formulate a stability theorem for fixed points of Courant algebroid structures on a given vector bundle E with fixed non-degenerate metric $\langle -, - \rangle$ (Theorem 3.5.36).

Finally, in the last subsection we consider Dirac structures inside a split Courant algebroid $A \oplus A^*$. Under the assumption that both A and A^* are Dirac structures, we apply the main theorem to obtain a sufficient condition for fixed points of the Dirac structure A. There is a difference from all the results obtained so far: the theorem does not only guarantee a fixed point near the given one, but in fact guarantees that the fixed point will lie on the same A^*-leaf as the original one (Theorem 3.5.50).

3.5.1 Higher order fixed points of Lie bialgebroids

Throughout this section, let (A, d_A) be a fixed Lie algebroid over the manifold M, with anchor ρ_A and bracket $[-, -]_A$. We apply Theorem 3.3.20, and find a sufficient condition for when a fixed point of (A^*, d_{A^*}) is stable for nearby Lie algebroid structures on A^* which are compatible with d_A, in a sense we make precise below.

3.5.1.1 Lie bialgebroids

We embed the Lie algebra of vector fields on $A^*[1]$ into a bigger graded Lie algebra, in which compatibility with the Lie algebroid structure on A can be formulated as a commutation condition. The details of this procedure can be found in Section 3 of [Roy99], but we describe the outline.

Given the graded manifold $A^*[1]$, we can consider the graded manifold $T^*[2]A^*[1]$. As ordinary cotangent bundles, this graded manifold carries a symplectic form, which has degree 2 in this case. To avoid going into details about this, we will work with the corresponding *dual* structure, which is a 2-Poisson algebra structure on $C^\infty(T^*[2]A^*[1])$ as in [CFL06] and can be described explicitly by its properties. The additional property this Poisson bracket has as it comes from a symplectic form, is that there is a Darboux-like theorem for the Poisson bracket [Cue21].
We summarise the properties of this graded manifold that we will use in the following proposition:

Proposition 3.5.1 ([Roy99, AN13, Cue21])**.** *The 2-Poisson algebra*

$$C^\infty(T^*[2]A^*[1])$$

satisfies the following properties.

i) $C^\infty(T^*[2]A^*[1])$ *is a* $(\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0})$*-bigraded algebra: as*

$$C^\infty(T^*[2]A^*[1]) = S_{C^\infty(A^*[1])}(\mathfrak{X}(A^*[1])[-2]),$$

the bidegree (p, q)*-part is given by*

$$S^p_{C^\infty(A^*[1])}(\mathfrak{X}(A^*[1])[-2])^{p+q}$$

ii) The Poisson bracket is non-degenerate. Moreover, the Poisson bracket is homogeneous of bidegree $(-1, -1)$ *with respect to the bigrading.*

iii) The Poisson bracket extends the bracket of vector fields on $A^*[-1]$*: any* $\delta \in \mathfrak{X}(A^*[1])$ *gives rise to* f_δ *of bidegree* $(1, |\delta| + 1)$*, compatible with the Lie bracket.*

iv) *The Lie algebroid structure d_A on A gives rise to an element $\Pi_{d_A} \in C^\infty(T^*[2]A^*[1])$ of bidegree $(2,1)$ as follows: as f_{d_A} defines a self-commuting element of $C^\infty(T^*[2]A[1])$ and there is a canonical symplectomorphism $T^*[2]A^*[1] \cong T^*[2]A[1]$, we get an element $\Pi_{d_A} \in C^\infty(T^*[2]A^*[1])$ satisfying $\{\Pi_A, \Pi_A\} = 0$.*

v) *The pair $((A, d_A), (A^*, d_{A^*}))$ is a Lie bialgebroid if and only if $\{f_{d_{A^*}}, \Pi_{d_A}\} = 0$.*

vi) *The action of d_{A^*} on $g \in C^\infty(A^*[1]) = \Gamma(S(A[-1]))$ is given by*

$$d_{A^*}(g) = \{f_{d_{A^*}}, g\}.$$

vii) *Π_{d_A} encodes the Schouten-Nijenhuis extension of the bracket on $\Gamma(A)$ to $C^\infty(A^*[1])$: given two homogeneous functions $f, g \in C^\infty(A^*[1])$,*

$$[f, g]_A = (-1)^{|f|-1}\{\{\Pi_{d_A}, f\}, g\}.$$

Remark 3.5.2. In classical terms, the compatibility condition between d_{A^*} and d_A can be expressed as follows: for $f, g \in C^\infty(M)$, $X, Y \in \Gamma(A)$,

$$[d_{A^*}(f), g]_A = [f, d_{A^*}(g)]_A,$$

$$d_{A^*}[X, f]_A = [d_{A^*}(X), f]_A + [X, d_{A^*}(f)]_A,$$

$$d_{A^*}[X, Y]_A = [d_{A^*}(X), Y]_A + [X, d_{A^*}(Y)]_A,$$

as a consequence of the Jacobi identity for $\{-,-\}$.

3.5.1.2 The ingredients

We check that we are in the setting of assumptions 3.3.17. Fix the Lie algebroid (A, d_A).

i) Proposition 3.5.1 now tells us that if we set

$$\mathfrak{g}_{LbA} := C^\infty(T^*[2]A^*[1])^{(\geq 1, \geq 1)}[1,1],$$

we have a differential bigraded Lie algebra, with differential $\{\Pi_{d_A}, -\}$ of bidegree $(1,0)$. Lie algebroid structures d_{A^*} on A^* such that $((A, d_A), (A^*, d_{A^*}))$ is a Lie bialgebroid are Maurer-Cartan elements of bidegree $(0,1)$. Note that because of the bigrading, the Maurer-Cartan equation for $f_{d_{A^*}} \in C^\infty(T^*[2]A^*[1])[1,1]$

$$\{\Pi_{d_A}, f_{d_{A^*}}\} + \frac{1}{2}\{f_{d_{A^*}}, f_{d_{A^*}}\} = 0$$

breaks up into two components: the bidegree $(1,1)$-part, which is

$$\{\Pi_{d_A}, f_{d_{A^*}}\} = 0,$$

and the bidegree $(0,2)$-part, which is

$$\{f_{d_{A^*}}, f_{d_{A^*}}\} = 0.$$

We would like to apply the main theorem in this setting, but as it is formulated, it is not clear it can be applied, as we are not interested in general Maurer-Cartan elements, but only those of bidegree $(0,1)$. The naive thing to do here would be to take the graded Lie algebra $\mathfrak{g}_{LbA}$ by forgetting the bigrading and only caring about the total degree (where we restrict to those where the unshifted bidegree is at least $(1,1)$). We show that this works.

ii) Let $p \in M$. We now define the Lie subalgebra $\mathfrak{g}_{LbA}(p,k)$ corresponding to fixed points of order $k \geq 0$. In bidegrees $(0,i)$ for $i = 0,1,2$ we see that $\mathfrak{g}^{(0,i)}_{LbA} = \mathfrak{X}(A^*[1])^i$. So we set

$$\mathfrak{g}^{(0,i)}_{LbA}(p,k) := \mathfrak{X}(A^*[1])^i_{p,i(k-1)+1},$$

as in Section 3.4.1. The compatibility with the Lie algebroid structure on A will be an extra condition on the cocycles in the quotient complex. Now set $\mathfrak{g}^{(1,0)}_{LbA}(p,k) = \mathfrak{g}^{(1,0)}_{LbA}, \mathfrak{g}^{(2,0)}_{LbA}(p,k) = \mathfrak{g}^{(2,0)}_{LbA}$, and

$$\mathfrak{g}^{(1,1)}_{LbA}(p,k) = \Big\{\Pi \in \mathfrak{g}^{(1,1)}_{LbA} \mid \{\{\Pi,\ f\}, g\} \in I_p^{k-j} C^\infty(A^*[1])^j$$
$$\forall f, g \in C^\infty(A^*[1])^{\leq 1}, j = |f| + |g|\Big\}.$$

This defines a differential graded Lie subalgebra:

Lemma 3.5.3. *For $i = 0,1,2$, let $\mathfrak{g}^i_{LbA}(p,k) = \bigoplus_{j=0}^i \mathfrak{g}^{(j,i-j)}_{LbA}(p,k)$. Then*

$$\{\mathfrak{g}^0_{LbA}(p,k), \mathfrak{g}^i_{LbA}(p,k)\} \subseteq \mathfrak{g}^i_{LbA}(p,k),$$

$$\{\mathfrak{g}^1_{LbA}(p,k), \mathfrak{g}^1_{LbA}(p,k)\} \subseteq \mathfrak{g}^2_{LbA}(p,k).$$

Remark 3.5.4. The only space which is unknown to us at this point is the $\mathfrak{g}^{(1,1)}_{LbA}/\mathfrak{g}^{(1,1)}_{LbA}(p,k)$. By picking an open neighborhood of $p \in M$ over which A trivializes, we see that

$$\mathfrak{g}^{(1,1)}_{LbA}/\mathfrak{g}^{(1,1)}_{LbA}(p,k) \cong J_p^{k-1}(S^2(TM)) \oplus J_p^{k-2}(\mathrm{Hom}(A \otimes T^*M, A))$$
$$\oplus J_p^{k-3}(\mathrm{Hom}(S^2(A[1]), S^2(A[1]))).$$

iii) The splittings exist for the same reason as for Lie algebroids.

iv) Now pick a Lie algebroid structure d_{A^*} on A^* such that $((A, d_A), (A^*, d_{A^*}))$ is a Lie bialgebroid structure, and some $p \in M$ is a fixed point of order k.

We now check that the data satisfies the assumptions 3.3.17.

a) We pick the following topologies:

   - On $\mathfrak{g}^0_{LbA}$, we pick the C^∞-topology,
   - On $\mathfrak{g}^1_{LbA}$, we pick the C^{2k-1}-topology,
   - On $\mathfrak{g}^2_{LbA}$, we pick the C^{2k-2}-topology.

b) As the $(2k-2)$-jet of $\{\Pi_A, X\}$ depends linearly on the $(2k-1)$-jet of X, $\{\Pi_A, -\}$ is continuous.

c) The continuity of $[-,-] : \mathfrak{g}^1_{LbA} \times \mathfrak{g}^1_{LbA} \to \mathfrak{g}^2_{LbA}$ holds for the same reasons as in 3.4.1.

d) It remains to understand the gauge action. It is clear that $\mathfrak{g}^{(0,0)}_{LbA} \cong CDO(A)$. Take $X \in \mathfrak{g}^{(0,0)}_{LbA}$ and $(Q, \Pi) \in \mathfrak{g}^1_{LbA} = \mathfrak{g}^{(0,1)}_{LbA} \oplus \mathfrak{g}^{(1,0)}_{LbA}$.
The gauge equation then becomes

$$\frac{d}{dt}(Q_t, \Pi_t) = (\{X, Q_t\}, \{X, \Pi_t\} - \{\Pi_A, X\}), \quad (Q_0, \Pi_0) = (Q, \Pi).$$

For the purposes of interpreting the main theorem it is sufficient to consider only the first component: as $\mathfrak{g}^{(1,0)}_{LbA}(p,k) = \mathfrak{g}^{(1,0)}_{LbA}$ the second component of the conclusion holds no information. The first component however, is simply the same as the gauge action for ordinary Lie algebroids: the take-away message is that

$$(Q_t, \Pi_t) \in \mathfrak{g}^1_{LbA}(p,k) \iff \phi^{\sigma(X)}_t(p) \text{ is a fixed point of } Q_t.$$

Here $\sigma(X) \in \mathfrak{X}(M)$ is the symbol of the differential operator X. Consequently, an open neighborhood of $0 \in \mathfrak{g}^0_{LbA}/\mathfrak{g}^0_{LbA}(p,k) \cong T_pM$ corresponds to an open neighborhood of $p \in M$.

e) Lemma 3.3.19 implies that the gauge action preserves Maurer-Cartan elements.

Remark 3.5.5.

i) When $k = 1$, the cohomology group

$$H^1\left(\mathfrak{g}_{LbA}/\mathfrak{g}_{LbA}(p,1), \overline{\{\Pi_{d_A}, -\} + \{f_{d_{A^*}}\}}\right)$$

is a subspace of $H^1_{CE}(A^*_p, T_pM)$, which is the cohomology group appearing in Theorem 3.2.8: the coboundaries remain unchanged, while the cocycles are the Chevalley-Eilenberg cocycles $\alpha \in A_p \otimes T_pM$, for which $(\rho_{A,p} \otimes \mathrm{id})(\alpha) \in T_pM \otimes T_pM$ is skew-symmetric. For general $k \geq 1$, there is an injective map

$$H^1\left(\mathfrak{g}_{LbA}/\mathfrak{g}_{LbA}(p,k), \ \overline{\{\Pi_{d_A}, -\}. + \{f_{d_{A^*}}, -\}}\right)$$
$$\hookrightarrow H^1\left(\mathfrak{g}_{LA}/\mathfrak{g}_{LA}(p,k), \overline{\{d_{A^*,} -\}}\right),$$

but for $k \geq 1$ the image is not as simple to describe. This is what we should expect: indeed, if a fixed point is stable for *all* Lie algebroid structures, it should in particular be stable for a subclass of Lie algebroid structures.

ii) The approach taken here to obtain the right graded Lie algebra might seem indirect, and a seemingly more direct approach would be to replace $(C^\infty(T^*[2]A^*[1]))^{(1,2)}$ by the kernel of the vertical differential $\{\Pi_{d_A}, -\}$. In order to make sure this is well-defined, one would also have to restrict the functions in bidegree (1,1) to those which are in the kernel of the vertical differential. As these are not given by the sections of some vector bundle over M in general, we would have less control over the objects we work with.

3.5.1.3 Applying the main theorem

Applying the main theorem to $\mathfrak{g} = \mathfrak{g}_{LbA}$, $\mathfrak{h} = \mathfrak{g}_{LbA}(p,k)$ yields:

Theorem 3.5.6. *Let (A, d_A) be a Lie algebroid over M, and (A^*, d_{A^*}) a Lie algebroid defined on the dual vector bundle such that $((A, d_A), (A^*, d_{A^*}))$ is a Lie bialgebroid. Let $p \in M$ be a fixed point of order k for $k \geq 0$ of d_{A^*}, that is, $f_{d_{A^*}} \in \mathfrak{g}^{(0,1)}_{LbA}(p,k)$. Assume that*

$$H^1\left(\mathfrak{g}_{LbA}/\mathfrak{g}_{LbA}(p,k), \overline{\{\Pi_{d_A}, -\} + \{f_{d_{A^*}}, -\}}\right) = 0,$$

where Π_{d_A}, $f_{d_{A^*}}$ are as in Proposition 3.5.1. Then for every open neighborhood U of $p \in M$, there exists a C^{2k-1}-neighborhood $\mathcal{U}$ of $d_{A^*} \in \mathfrak{g}^1_{LbA}$ such that for any Lie algebroid structure $Q \in \mathcal{U}$ compatible with d_A, there is a family I in U of fixed points of order k of Q parametrized by an open neighborhood of

$$0 \in H^0(\mathfrak{g}_{LbA}/\mathfrak{g}_{LbA}(p,k), \overline{\{\Pi_{d_A}, -\} + \{f_{d_{A^*}}, -\}}).$$

3.5.1.4 Poisson manifolds as Lie bialgebroid structures

In this section we apply Theorem 3.5.6 to the case where $A = TM$ with its standard Lie algebroid structure $d_A = d_{dR}$. In this case, Lie algebroid structures on T^*M compatible with d_{dR} are in bijection with Poisson structures, as was pointed out in [Roy02, Corollary 5.3]. We briefly sketch the correspondence.

Lemma 3.5.7. *Let $A = TM$, with $d_A = d_{dR}$ being the standard Lie algebroid structure on TM. Then a Lie algebroid structure d_{T^*M} on T^*M such that $((TM, d_{dR}), (T^*M, d_{T^*M}))$ is a Lie bialgebroid structure is equivalent to a Poisson structure on M.*

Sketch of proof. Given a Poisson structure $\pi \in \mathfrak{X}^2(M)$, the usual Lie algebroid structure on T^*M is compatible with the de Rham differential. Conversely, given any Lie algebroid structure d_{T^*M} on T^*M compatible with d_{dR}, the map

$$\pi_{d_{T^*M}} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

given by

$$\pi_{d_{T^*M}}(f, g) := [d_{T^*M}(f), g]$$

is a Poisson structure for which the induced Lie algebroid structure on T^*M is d_{T^*M}. Here the bracket on the right hand side is the usual Schouten-Nijenhuis bracket. ☐

In particular, this implies that Theorem 3.5.6 gives a result for Poisson manifolds. The rest of this section is dedicated to showing that this is in fact equivalent to the result obtained in [DW06]. For this we show that the vanishing of the cohomologies are equivalent conditions.

Given a Poisson structure π on M such that its $k-1$-jet at $p \in M$ vanishes, consider the graded Lie subalgebra given by

$$\bigoplus_{j=1}^{n} I_p^{1+j(k-1)} \mathfrak{X}^j(M),$$

where $n = \dim(M)$. Using the reasoning from Section 3.4.1.3, one can show that the cohomological assumption in [DW06, Theorem 1.2] can be restated as follows.

Lemma 3.5.8. *The vanishing of the cohomology in the hypothesis of [DW06, Theorem 1.2] is equivalent to*

$$H^2(\mathfrak{X}^\bullet(M)/I_p^{1+(\bullet-1)(k-1)}\mathfrak{X}^\bullet(M), \overline{[\pi, -]}) = 0.$$

The cohomology $H^1(\mathfrak{g}_{LbA}/\mathfrak{g}_{LbA}(p,k), \overline{\{\Pi_{d_{dR}}, -\} + \{f_{[\pi,-]}, -\}})$ from Theorem 3.5.6 when applied to $A = TM$ with the standard Lie algebroid structure is isomorphic to the cohomology stated in the lemma:

Proposition 3.5.9. *Let $\pi \in \mathfrak{X}^2(M)$ be as above. The injective differential graded Lie algebra map $H : (\mathfrak{X}^\bullet(M)[1], [\pi, -]) \to (\mathfrak{g}^\bullet_{LbA}, \{\Pi_{d_{dR}}, -\} + \{f_{[\pi,-]}, -\})$*

$$
\begin{array}{ccc}
\mathfrak{X}(M) & \xrightarrow{\quad f_{[\bullet,-]} \quad} & \mathfrak{g}^0_{LbA} \\
\Big\downarrow [\pi,-] & & \Big\downarrow \begin{pmatrix} \{f_{[\pi,-]},-\} \\ \{\Pi_{d_{dR}},-\} \end{pmatrix} \\
\mathfrak{X}^2(M) & \xrightarrow{\begin{pmatrix} f_{[\bullet,-]} \\ 0 \end{pmatrix}} & \mathfrak{g}^{(0,1)}_{LbA} \oplus \mathfrak{g}^{(1,0)}_{LbA} \\
\Big\downarrow [\pi,-] & & \Big\downarrow \begin{pmatrix} \{f_{[\pi,-]},-\} & 0 \\ \{\Pi_{d_{dR}},-\} & \{f_{[\pi,-]},-\} \\ 0 & \{\Pi_{d_{dR}},-\} \end{pmatrix} \\
\mathfrak{X}^3(M) & \xrightarrow{\begin{pmatrix} f_{[\bullet,-]} \\ 0 \\ 0 \end{pmatrix}} & \mathfrak{g}^{(0,2)}_{LbA} \oplus \mathfrak{g}^{(1,1)}_{LbA} \oplus \mathfrak{g}^{(2,0)}_{LbA}
\end{array}
$$

descends to an injective chain map:

$$
\begin{array}{ccc}
\mathfrak{X}(M)/I_p\mathfrak{X}(M) & \xrightarrow{\quad \overline{f_{[\bullet,-]}} \quad} & \mathfrak{g}^0_{LbA}/\mathfrak{g}^0_{LbA}(p,k) \\
\Big\downarrow \overline{[\pi,-]} & & \Big\downarrow \overline{\{f_{[\pi,-]},-\}} \\
\mathfrak{X}^2(M)/I_p^k\mathfrak{X}^2(M) & \xrightarrow{\quad \overline{f_{[\bullet,-]}} \quad} & \mathfrak{g}^{(0,1)}_{LbA}/\mathfrak{g}^{(0,1)}_{LbA}(p,k) \\
\Big\downarrow \overline{[\pi,-]} & & \Big\downarrow \begin{pmatrix} \overline{\{f_{[\pi,-]},-\}} \\ \overline{\{\Pi_{d_{dR}},-\}} \end{pmatrix} \\
\mathfrak{X}^3(M)/I_p^{2k-1}\mathfrak{X}^3(M) & \xrightarrow{\begin{pmatrix} \overline{f_{[\bullet,-]}} \\ 0 \end{pmatrix}} & \mathfrak{g}^{(0,2)}_{LbA}/\mathfrak{g}^{(0,2)}_{LbA}(p,k) \oplus \mathfrak{g}^{(1,1)}_{LbA}/\mathfrak{g}^{(1,1)}_{LbA}(p,k)
\end{array}
\quad . \tag{3.18}
$$

Moreover, the top row of (3.18) *is an isomorphism and the middle row is an isomorphism when restricted to cocycles. Consequently, the induced map on middle cohomology is an isomorphism.*

Proof. It is straightforward to see that the map descends to an injective chain map on the quotients.
Note that both spaces in the top row of (3.18) can be identified with T_pM, and that the map is compatible with this identification.

As the middle map is injective and it preserves the cocycles, it is sufficient to show that it is surjective on cocycles. Let $f_\delta + \mathfrak{g}^{(0,1)}_{LbA}(p,k)$ be a cocycle on the right hand side for some $\delta \in \mathfrak{X}^1(T^*M[1])$. Motivated by Lemma 3.5.7, define the bivector field on M given by

$$\pi_\delta(f,g) := \frac{1}{2}([\delta(f),g] - [\delta(g),f])$$

for $f,g \in C^\infty(M)$.
Using the second component of the cocycle condition in (3.18) stating that $\{\Pi_{d_{dR}}, f_\delta\} \in \mathfrak{g}^{(1,1)}_{LbA}(p,k)$, it follows that

$$f_\delta - f_{[\pi_\delta,-]} \in \mathfrak{g}^{(0,1)}_{LbA}(p,k),$$

so the class of f_δ lies in the image of $\overline{H_1}$. It remains to show that $\pi_\delta + I_p^k\mathfrak{X}^2(M)$ is a cocycle. As

$$\{f_{[\pi,-]}, f_{[\pi_\delta,-]}\} = f_{[[\pi,\pi_\delta],-]} \in \mathfrak{g}^{(0,2)}_{LbA}(p,k).$$

Using injectivity of H, it follows that $[\pi,\pi_\delta] \in I_p^{2k-1}\mathfrak{X}^3(M)$, concluding the proof. ☐

We therefore obtain:

Corollary 3.5.10. *Theorem 3.5.6 applied to* $(A, d_A) = (TM, d_{dR})$ *is equivalent to [DW06, Theorem 1.2].*

3.5.1.5 Poisson manifolds with a Poisson hypersurface as Lie bialgebroid structures

Here we apply Theorem 3.5.6 to the case where $A = {}^bTM$. Let M be a smooth manifold, and let $Z \subseteq M$ be a smooth hypersurface. Denote by $A = {}^bTM$ the b-tangent bundle, with its standard Lie algebroid structure d_A (see [GMP14] for details). Its sections are defined by

$$\Gamma(A) := \{X \in \mathfrak{X}(M) \mid X|_Z \in \mathfrak{X}(Z)\},$$

the anchor is the inclusion, which uniquely determines the bracket. For M a manifold with boundary, the b-tangent bundle for $Z = \partial M$ was introduced in [Mel93].

For $A = {}^bTM$ with the Lie algebroid structure as described above, we can characterize Lie algebroid structures d_{A^*} on A^* such that $((A, d_A), (A^*, d_{A^*}))$ is a Lie bialgebroid explicitly, and give a more direct description of the relevant cohomology. The analogue of Lemma 3.5.7 holds in this setting, of which we omit the proof.

Lemma 3.5.11. *Let $A = {}^{\flat}TM$ with its standard Lie algebroid structure d_A. A Lie algebroid structure d_{A^*} on A^* such that $((A, d_A), (A^*, d_{A^*}))$ is a Lie bialgebroid is the same as a self-commuting section $\pi_{d_{A^*}} \in \Gamma(\wedge^2 A)$. Moreover, self-commuting $\pi \in \Gamma(\wedge^2 A)$ are in bijection with Poisson structures on M, such that Z is a Poisson submanifold.*

Using this, Theorem 3.5.6 implies the following.

Theorem 3.5.12. *Let (M, Z) be a manifold with a given hypersurface Z. Let $A = {}^{\flat}TM$ be the b-tangent bundle with its standard Lie algebroid structure. Let $k \geq 1$ be an integer, and let $\pi \in \Gamma(\wedge^2 A)$ be a self-commuting element. Let $p \in M$ be a fixed point of order k of $[\pi, -]$, which is the Lie algebroid structure on A^*, and assume that*

$$H^1\left(\mathfrak{g}_{LbA}/\mathfrak{g}_{LbA}(p,k), \overline{\{\Pi_{d_A}, -\} + \{f_{d_{A^*}}, -\}}\right) = 0.$$

Then for every neighborhood U of p, there is a C^{2k-1}-neighborhood $\mathcal{U}$ of π such that for every self-commuting $\pi' \in \mathcal{U}$ there is a family I in U of fixed points of order k of $[\pi', -]$ parametrized by an open neighborhood of

$$0 \in H^0\left(\mathfrak{g}_{LbA}/\mathfrak{g}_{LbA}(p,k), \overline{\{\Pi_{d_A}, -\} + \{f_{d_{A^*}}, -\}}\right).$$

Next, we spell out what the requirement that some point $p \in M$ is a fixed point of order k of the Lie algebroid structure $[\pi, -]$ means directly in terms of π, describe the relevant cohomology in more detail for fixed points order 1, and improve the result in this case using remark 3.3.21ix).
When considering fixed points $p \in M$ of the corresponding Lie algebroid structure on A^*, we distinguish two types:

- $p \in Z$,
- $p \in M \setminus Z$.

The second case does not yield anything new: as the problem is local, and $A|_{M\setminus Z} = T(M\setminus Z)$ as Lie algebroids, we recover the result for Poisson structures of the previous section.
The following lemma describes what the notion of fixed point of order k of $[\pi, -]$ implies about π.

Lemma 3.5.13. *Let $\pi \in \Gamma(S^2(A[-1]))$, and $p \in Z \subseteq M$. Then $[\pi, -] \in \mathfrak{g}_{LbA}^{(0,1)}(p,k)$ if and only if $\pi \in I_p^k\Gamma(S^2(A[-1]))$.*

Moreover, in this case the Poisson bracket $\{-,-\}_\pi$ *on* $C^\infty(M)$ *induced by applying* ρ_A *to* π *satisfies*

$$\{I_Z, C^\infty(M)\}_\pi \subseteq I_Z \cdot I_p^k,$$

$$\{C^\infty(M), C^\infty(M)\}_\pi \subseteq I_p^k.$$

Proof. As the statement is local, we assume that $M = \mathbb{R}^n$, and $Z = \{x^1 = 0\}$. Let $\{e_i\}_{i=1}^n$ be the induced frame on A, with

$$\rho(e_i) := \begin{cases} x^1 \partial_{x^1} & i = 1, \\ \partial_{x^i} & i \neq 1. \end{cases}$$

Write in terms of this frame

$$\pi = \frac{\pi^{ji}}{2} e_i \cdot e_j,$$

where $\cdot$ is the shifted symmetric product and $\pi^{ij} = -\pi^{ji} \in C^\infty(M)$.
Assume that $[\pi, -] \in \mathfrak{g}_{LbA}^{(0,1)}(p,k)$. This implies that for all functions $f \in C^\infty(M)$, we have

$$\begin{aligned}[\pi, f] &= \frac{\pi^{ji}}{2}[e_i \cdot e_j, f] \\ &= \frac{\pi^{ji}}{2}(\rho_A(e_j)(f)e_i - \rho_A(e_i)(f)e_j) \\ &= \sum_{i=2}^n \pi^{1i}(\partial_{x^i}(f)e_1 - x^1\partial_{x^1}(f)e_i) + \sum_{i,j\geq 2} \frac{\pi^{ji}}{2}(\partial_{x^j}(f)e_i - \partial_{x^i}(f)e_j) \in I_p^k\Gamma(A),\end{aligned}$$

As this holds for all $f \in C^\infty(M)$, we get $\pi^{ji} \in I_p^k$ for all i, j.
Conversely, if $\pi \in I_p^k\Gamma(S^2(A[-1]))$, the above computation shows that $[\pi, f] \in I_p^k\Gamma(A)$ for all $f \in C^\infty(M)$. Moreover, for $e_i \in \Gamma(A)$,

$$[\pi, e_l] = \begin{cases} -x^1 \frac{\partial_{x^1}(\pi^{ji})}{2} e_i \cdot e_j & l = 1, \\ -\frac{\partial_{x^l}(\pi^{ji})}{2} e_i \cdot e_j & l \neq 2, \end{cases}$$

from which it follows that $[\pi, e_i] \in I_p^{k-1}\Gamma(S^2(A[-1]))$, showing that $[\pi, -] \in \mathfrak{g}_{LbA}^{(0,1)}(p,k)$.
The assertion about the induced Poisson bracket on $C^\infty(M)$ follows from the above computation. □

For the rest of the section, assume that $p \in Z$ is a fixed point of order 1 of the Lie algebroid structure $(A^*, [\pi, -]_A)$ for some self-commuting $\pi \in \Gamma(S^2(A[-1]))$. In this case, we can give a more explicit description of the cohomology appearing in the theorem. As the problem is local around $p \in Z \subseteq M$, we assume that $M = \mathbb{R}^n$, $p = 0$, and $Z = \{x^1 = 0\}$. Consider the induced frame $\{e_i\}_{i=1}^n$ for A as in the proof of Lemma 3.5.13, with dual frame $\{e^i\}_{i=1}^n$. Following remark 3.5.5i) the cochain spaces are given as follows. In degree 0, we have T_pM, while in degree 1, we can restrict ourselves to the span of

$$\{e_1 \otimes \partial_{x^j} \in A_p \otimes T_pM \mid i = 1, \dots, n\} \cup \{e_i \otimes \partial_{x^j} - e_j \otimes \partial_{x^i} \mid 2 \leq i, j \leq n\} \subseteq A_p \otimes T_pM,$$

by the skew-symmetry requirement. Finally, in degree 2, we have $S^2(A_p[-1]) \otimes T_pM$. The differentials are given by the Chevalley-Eilenberg formulas as in Definition 3.2.5.

Example 3.5.14. Let $M = \mathbb{R}^3$, with coordinates (x, y, z), and $Z = \{x = 0\}$. Let

$$\pi = xe_2 \wedge e_3 + ye_3 \wedge e_1 + ze_1 \wedge e_2.$$

It is easy to see that $[\pi, \pi] = 0$. The corresponding Lie algebroid structure on A^* has a fixed point of order 1 in the origin p. By the discussion above, to compute the cohomology in degree 1, it is sufficient to restrict ourselves to the subspace of $A_p \otimes T_pM$ generated by $\{e_1 \otimes \partial_x, e_1 \otimes \partial_y, e_1 \otimes \partial_z, e_2 \otimes \partial_z - e_3 \otimes \partial_y\}$. One can show that the cohomology

$$H^1\left(\mathfrak{g}_{LbA}/\mathfrak{g}_{LbA}(p,k), \overline{\{\Pi_A, -\} + \{f_{d_{A^*}}, -\}}\right),$$

as described in remark 3.5.5 vanishes, and hence the fixed point is stable.

The condition that

$$H^1\left(\mathfrak{g}_{LbA}/\mathfrak{g}_{LbA}(p,k), \overline{\{\Pi_A, -\} + \{f_{d_{A^*}}, -\}}\right) = 0$$

is only a sufficient condition, but not a necessary one, as the following example shows.

Example 3.5.15. For $M = \mathbb{R}^2$, $Z = \{x = 0\}$, consider $\pi = f(x, y)e_1 \wedge e_2$ for some $f \in C^\infty(M)$, with $f(p) = 0$, $df_p \neq 0$. The equation $[\pi, \pi] = 0$ is trivially satisfied. As $0 \in \mathbb{R}$ is a regular value of f, we know that p is a stable fixed point of order 1.
We now compute the relevant cohomology. By the description above, we can restrict ourselves to the span of $e_1 \otimes \partial_x$, $e_1 \otimes \partial_y$.
Denote the corresponding differential by d_π. Then

$$d_\pi(e_1 \otimes \partial_x) = f(p)\partial_x = 0,$$

and

$$d_\pi(e_1 \otimes \partial_y) = -x\partial_x f(x,y)\big|_{(x,y)=p}\,\partial_y = 0.$$

So both elements are cocycles. To see what the coboundaries are, we compute d_π of T_pM.

$$d_\pi(\partial_x) = -\partial_x f(p) e_1 \otimes \partial_y,$$
$$d_\pi(\partial_y) = -\partial_y f(p) e_1 \otimes \partial_y,$$

showing that $e_1 \otimes \partial_x$ is never a coboundary, while $e_1 \otimes \partial_y$ is always a coboundary. Consequently, the relevant cohomology never vanishes when $\dim M = 2$.

More generally, it is true in every dimension that $e_1 \otimes \partial_{x^1}$ is never a coboundary. In fact, looking at the proof of Lemma 3.5.13, we see that when we take any $\pi \in \Gamma(S^2(A[-1]))$, the class of

$$[\pi, -] \in \mathfrak{g}^{(0,1)}_{LbA}/\mathfrak{g}^{(0,1)}_{LbA}(p,1) \cong A_p \otimes T_pM$$

takes values in the subspace spanned by

$$K = \{e_i \otimes \partial_j \in A_p \otimes T_pM \mid (i,j) \neq (1,1)\}. \tag{3.19}$$

Note that this is an instance of remark 3.3.21ix), hence we can improve the result by putting a milder restriction on the relevant cohomology, by replacing $A_p \otimes T_pM$ by K in the proof of Theorem 3.5.6 for $k = 1$. Fix coordinates near p as in the proof of Lemma 3.5.13.

Theorem 3.5.16. *Let $\pi \in \Gamma(S^2(A[-1]))$, with $[\pi, \pi] = 0$. Assume $p \in M$ is such that $\pi_p = 0$. If*

$$H^1_{red} = 0,$$

as defined in remark 3.3.21ix) for K as in equation (3.19)*, the conclusion of Theorem 3.5.12 holds for $k = 1$.*

Remark 3.5.17. Note that Theorem 3.5.16 does not guarantee the existence of a fixed point in Z. We will revisit this in example 3.5.52.

3.5.1.6 Poisson-Nijenhuis structures as Lie bialgebroids

Another class of Lie bialgebroids comes from Poisson-Nijenhuis structures as shown in [KS96]. We will give two results regarding stability of fixed points of Poisson structures, compatible with a given Nijenhuis tensor N. Theorem 3.5.20 deals with stability for nearby Lie algebroid structures on T^*M compatible with N in the Lie bialgebroid sense, while Theorem 3.5.21 deals with nearby Poisson structures which form a Poisson-Nijenhuis pair with N.

Classically, Poisson-Nijenhuis structures are the following.

Definition 3.5.18 ([KSM90])**.** Let M be a manifold. A *Poisson-Nijenhuis structure* is a pair (π, N), where

i) $\pi \in \mathfrak{X}^2(M)$,

ii) $N \in \Gamma(\mathrm{End}(TM))$,

satisfying

a) $[\pi, \pi] = 0$,

b) $[N, N]_{FN} = 0$,

c) $\pi^\sharp \circ N^* = N \circ \pi^\sharp$,

d) $[\alpha, \beta]_{N\pi} = [N^*\alpha, \beta]_\pi + [\alpha, N^*\beta]_\pi - N^*[\alpha, \beta]_\pi$ for all $\alpha, \beta \in \Omega^1(M)$.

In the definition above, $[-,-]_{FN}$ is the Frölicher-Nijenhuis bracket, and for a bivector field π, the bracket $[-,-]_\pi$ is the one induced on $\Omega^1(M)$. Note that $N\pi$ is the bivector field defined by the equation $(N\pi)^\sharp = N \circ \pi^\sharp$, which is skew-symmetric by c).
Note that N^* can be extended as a derivation to $\Omega^\bullet(M)$, the graded algebra of all differential forms on M. As shown in [KSM90], condition b) implies that

$$[d, N^*] \in \mathfrak{X}^1(TM[1])$$

is a Lie algebroid structure on TM. Denote this Lie algebroid by (A, d_N). By [KS96, Proposition 3.2], the compatiblity conditions between π, N can be expressed as follows.

Proposition 3.5.19 ([KS96])**.** *Let M be a manifold, $\pi \in \mathfrak{X}(M)$ and $N \in \Gamma(\mathrm{End}(TM))$. Then (π, N) is a Poisson-Nijenhuis structure if and only if $((A, d_N), (A^*, [\pi, -]))$ is a Lie bialgebroid.*

Theorem 3.5.6 now yields a stability criterium for fixed points of Lie algebroid structures on T^*M near a Poisson-Nijenhuis structure:

Theorem 3.5.20. *Let M be a manifold equipped with a Poisson-Nijenhuis structure (π, N) and $p \in M$ such that the $(k-1)$-jet of π vanishes at p. Assume that*

$$H^1\left(\mathfrak{g}_{LbA}/\mathfrak{g}_{LbA}(p,k), \overline{\{\Pi_{d_N}, -\} + \{f_{[\pi,-]}, -\}}\right) = 0,$$

where Π_{d_N} corresponds to the Lie algebroid structure d_N on TM induced by N. Then for every neighborhood $U \subseteq M$ of p, there is a C^{2k-1}-neighborhood $\mathcal{U}$ of

$[\pi, -] \in \mathfrak{X}^1(A^*[1])$ *such that for any Lie algebroid structure* $Q \in \mathcal{U}$ *such that* $((A, d_N), (A^*, Q))$ *is a Lie bialgebroid there is a family* I *in* U *of fixed points of order* k *of* Q *parametrized by a neighborhood of*

$$0 \in H^0\left(\mathfrak{g}_{LbA}/\mathfrak{g}_{LbA}(p,k), \overline{\{\Pi_{d_N}, -\} + \{f_{[\pi,-]}, -\}}\right).$$

Note that the conclusion of the theorem for Lie bialgebroids is stronger than the conclusion that all nearby π' such that (π', N) is a Poisson-Nijenhuis structure must have a fixed point of order k near p. For an arbitrary Nijenhuis tensor N, there might be Lie algebroid structures near $[\pi, -]$ which do not come from Poisson structures. However, by making different choices for $\mathfrak{g}$ and $\mathfrak{h}$ in the main theorem, we can obtain a result which deals with the problem of stability within the realm of Poisson-Nijenhuis structures.

One way to do this is to consider the Lie subalgebra $\mathfrak{g}_{PN}$ of $\mathfrak{g}_{LbA}$ given by the image of the inclusion

$$\mathfrak{X}^{i+1}(M) \hookrightarrow \mathfrak{g}_{LbA}^{(0,i)}$$

as in Section 3.5.1.4. For the subalgebra $\mathfrak{g}_{PN}(p,k)$ corresponding to fixed points, simply restrict to the intersection of $\mathfrak{g}_{LbA}^{(0,i)}(p,k)$ with the image of the inclusion above. The cochain spaces $\mathfrak{g}_{PN}/\mathfrak{g}_{PN}(p,k)$ then take the form

$$\mathfrak{g}^0_{PN}/\mathfrak{g}^0_{PN}(p,k) \cong T_pM,$$

$$\mathfrak{g}^1_{PN}/\mathfrak{g}^1_{PN}(p,k) \cong J_p^{k-1}(TM)$$

$$\mathfrak{g}^2_{PN}/\mathfrak{g}^2_{PN}(p,k) \cong J_p^{2k-2}(TM) \oplus \mathfrak{g}_{LbA}^{(1,1)}/\mathfrak{g}_{LbA}^{(1,1)}(p,k).$$

Applying Theorem 3.3.20 to this data, we find:

Theorem 3.5.21. *Let* M *be a manifold equipped with a Poisson-Nijenhuis structure* (π, N) *and* $p \in M$ *such that the* $(k-1)$*-jet of* π *vanishes at* p*. If*

$$H^1\left(\mathfrak{g}_{PN}/\mathfrak{g}_{PN}(p,k), \overline{\{\Pi_{d_N}, -\} + \{f_{[\pi,-]}, -\}}\right) = 0,$$

then for every neighborhood $U \subseteq M$ *of* p*, there is a* C^{2k-1}*-neighborhood* $\mathcal{U}$ *of* $\pi \in \mathfrak{X}^2(M)$ *such that for any Poisson structure* $\pi' \in \mathcal{U}$ *such that* (π', N) *is a Poisson-Nijenhuis structure there is a family* I *in* U *of fixed points of order* k *of* π' *parametrized by a neighborhood of*

$$0 \in H^0\left(\mathfrak{g}_{PN}/\mathfrak{g}_{PN}(p,k), \overline{\{\Pi_{d_N}, -\} + \{f_{[\pi,-]}, -\}}\right).$$

Remark 3.5.22.

- For $k = 1$, the cohomology can be described explicitly: unpacking the definition of

$$H^1\left(\mathfrak{g}_{PN}/\mathfrak{g}_{PN}(p,k), \overline{\{\Pi_{d_N}, -\} + \{f_{[\pi,-]}, -\}}\right)$$

shows that it is the cohomology of the complex

$$T_pM \xrightarrow{\overline{[\pi,-]}} S^2(T_pM[-1]) \xrightarrow{\begin{pmatrix} \overline{[\pi,-]} \\ \overline{d_N} \end{pmatrix}} S^3(T_pM[-1]) \oplus S^2(T_pM),$$

where for $\pi' \in S^2(T_pM[-1])$,

$$d_N(\pi') = (N\pi'^{\sharp})_{sym}$$

is the symmetric part of $N\pi'^{\sharp}$. To compute the middle cohomology, we may then restrict ourselves to those cocycles, for which $N\pi'^{\sharp}$ is still skew-symmetric.

- Note that by construction there is an injective map

$$H^1\left(\mathfrak{g}_{PN}/\mathfrak{g}_{PN}(p,k), \delta\right) \hookrightarrow H^1\left(\mathfrak{g}_{LbA}/\mathfrak{g}_{LbA}(p,k), \delta\right),$$

where $\delta = \overline{\{\Pi_{d_N}, -\} + \{f_{[\pi,-]}, -\}}$.

An important example of a Nijenhuis tensor is a complex structure on a manifold M: it is a map $J : TM \to TM$ with $J^2 = -\mathrm{id}$, such that $[J,J]_{FN} = 0$. It is shown in [LGSX08, Theorem 2.7] that a Poisson-Nijenhuis structure (π, J) is equivalent to $J\pi^\sharp + i\pi^\sharp \in \Gamma(\wedge^2 TM_{\mathbb{C}})$ being a holomorphic Poisson structure. Applying Theorem 3.5.21 yields:

Corollary 3.5.23. *If* $H^1\left(\mathfrak{g}_{PN}/\mathfrak{g}_{PN}(p,k), \overline{\{\Pi_{d_J}, -\} + \{f_{[\pi,-]}, -\}}\right) = 0$*, every holomorphic Poisson structure near* $J\pi^\# + i\pi^\#$ *has a family of fixed points of order* k *near* p*, parametrized by a neighborhood of*

$$0 \in H^0\left(\mathfrak{g}_{PN}/\mathfrak{g}_{PN}(p,k), \overline{\{\Pi_{d_J}, -\} + \{f_{[\pi,-]}, -\}}\right).$$

Remark 3.5.24. Note that if a holomorphic Poisson structure vanishes up to first order at a point ($k = 1$), the $(1,0)$-cotangent space at p of M inherits a complex Lie algebra structure $\mathfrak{g}_{\mathbb{C}}$. As pointed out by Marius Crainic, the cohomology

$$H^1\left(\mathfrak{g}_{PN}/\mathfrak{g}_{PN}(p,1), \overline{\{\Pi_{d_J}, -\} + \{f_{[\pi,-]}, -\}}\right)$$

is isomorphic to the *complex* Lie algebra cohomology

$$H^2_{CE}(\mathfrak{g}_{\mathbb{C}}, \mathbb{C}).$$

Example 3.5.25. We compute an explicit example of Corollary 3.5.23. Consider $M = \mathbb{R}^4 \cong \mathbb{C}^2$, with coordinates $(x^1, y^1, x^2, y^2) = (z^1, z^2)$, where

$$z^j = x^j + iy^j,$$

for $j = 1, 2$. On M consider the real Poisson structure

$$\pi = y^1(\partial_{x^1} \wedge \partial_{x^2} - \partial_{y^1} \wedge \partial_{y^2}) - x^1(\partial_{x^1} \wedge \partial_{y^2} + \partial_{y^1} \wedge \partial_{x^2}),$$

and let J be the standard complex structure induced by multiplication by i. Then (π, J) is a Poisson-Nijenhuis structure (π is the imaginary part of the holomorphic bivector field $z^1\partial_{z^1} \wedge \partial_{z^2}$), and π vanishes in the origin. In this case the relevant cohomology

$$H^1\left(\mathfrak{g}_{PN}/\mathfrak{g}_{PN}(p,k), \overline{\{\Pi_A, -\} + \{f_{[\pi,-]}, -\}}\right)$$

vanishes, and for nearby Poisson structures π' such that (π', J) is Poisson-Nijenhuis, there is a q near the origin such that π' vanishes in q. Note that this is really only the case for those Poisson structures for which (π', J) is Poisson-Nijenhuis: the bivector field

$$\pi_\epsilon = \pi + \epsilon\partial_{x^2} \wedge \partial_{y^2}$$

is Poisson, but non-vanishing.

3.5.2 Higher order fixed points of Courant algebroids

In this section we apply the main theorem to Courant algebroids, and obtain a stability result along the same lines as before.

3.5.2.1 Courant algebroids

Classically, a Courant algebroid is defined as follows. See e.g. [LBM09].

Definition 3.5.26. A *Courant algebroid* over a manifold M is a quadruple $(E, \langle -, - \rangle, \rho, [\![-, -]\!])$, where

i) E is a vector bundle over M,

ii) $\langle -, - \rangle : E \times E \to \mathbb{R}$ is a fiberwise symmetric, non-degenerate bilinear pairing,

iii) $\rho : E \to TM$ is a bundle map,

iv) $[\![-,-]\!] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ is an $\mathbb{R}$-bilinear map

satisfying for $x, y, z \in \Gamma(A)$

a) $[\![x, [\![y, z]\!]]\!] = [\![[\![x, y]\!], z]\!] + [\![y, [\![x, z]\!]]\!]$,

b) $\rho(x)\langle y, z\rangle = \langle [\![x, y]\!], z\rangle + \langle y, [\![x, z]\!]\rangle$,

c) $[\![x, y]\!] + [\![y, x]\!] = \rho^* d\langle x, y\rangle$, where $\rho^* : T^*M \to E^*$ is the dual map to ρ, and we identify $E^* \cong E$ via $\langle -, -\rangle$.

This definition implies among others the following property, as can be found in e.g. [LBM09].

Lemma 3.5.27. *The bracket $[\![-,-]\!]$ satisfies the Leibniz rule in the right entry, i.e. for $x, y \in \Gamma(E)$, $f \in C^\infty(M)$,*

$$[\![x, fy]\!] = f[\![x, y]\!] + \rho(x)(f)y.$$

Consequently, ρ is a morphism of brackets.

This means that a Courant algebroid gives rise to a foliation on M, and it makes sense to speak of fixed points of a Courant algebroid.
Due to D. Roytenberg [Roy02], there is a more concise definition, which is closer to our approach here, making use of graded geometry.
Fix a vector bundle E with a non-degenerate, symmetric bilinear pairing $\langle -, -\rangle$. Then:

Proposition 3.5.28 ([Roy02])**.** *There is a degree 2 graded manifold $\mathcal{M}_{E,\langle -,-\rangle}$ associated to the pseudo-Euclidean vector bundle $(E, \langle -, -\rangle)$, which is symplectic: its functions $C^\infty(\mathcal{M}_{E,\langle -,-\rangle})$ are equipped with a degree -2 Poisson bracket $\{-.-\}$, which is non-degenerate. Conversely, any symplectic degree 2 graded manifold arises in this way.*
Moreover, Courant algebroid structures on $(E, \langle -, -\rangle)$ are in 1-1 correspondence with functions $\Theta \in C^\infty(\mathcal{M}_{E,\langle -,-\rangle})^3$ satisfying $\{\Theta, \Theta\} = 0$.

3.5.2.2 The ingredients

We check that we are in the setting of assumptions 3.3.17.

i) An alternative description of a degree -2 Lie bracket is a degree 0 Lie bracket on the algebra $C^\infty(\mathcal{M}_{E,\langle -,-\rangle})[2]$. This is now a graded Lie algebra $\mathfrak{g}_{CA} := C^\infty(\mathcal{M}_{E,\langle -,-\rangle})[2]$ such that $\mathfrak{g}^1_{CA} = C^\infty(\mathcal{M}_{E,\langle -,-\rangle})^3$, and its Maurer-Cartan elements are precisely Courant algebroid structures on E. Intuitively, elements of $\mathfrak{g}^i_{CA} = C^\infty(\mathcal{M}_{E,\langle -,-\rangle})^{i+2}$ consist of graded symmetric products of elements of $\Gamma(E)$ and $\mathfrak{X}(M)$, the former counting as degree 1, while the latter counts as degree 2. This can be made precise by picking a connection ∇ on E respecting the pairing. Then we can identify

$$C^\infty(\mathcal{M}_{E,\langle -,-\rangle}) \cong \Gamma(S(E[-1] \oplus TM[-2])).$$

Here we implicitly use that $\langle -,-\rangle$ induces an isomorphism $E \cong E^*$.
In particular, every degree is given by the sections of some vector bundle.

ii) Similar to Lie algebroids, we define the notion of $p \in M$ being a fixed point of order $k \geq 0$ of a Courant algebroid structure, and show that there is a graded Lie subalgebra $\mathfrak{g}_{CA}(p,k)$, whose Maurer-Cartan elements are those Courant algebroid structures with for which $p \in M$ is a fixed point of order k.

Definition 3.5.29. Let $(E, \langle -,-\rangle)$ be a vector bundle with a symmetric non-degenerate pairing. Define for $p \in M$, $l \geq 0$:

$$C^\infty(\mathcal{M}_{E,\langle -,-\rangle})^2_{p,l} := \Big\{ X \in C^\infty(\mathcal{M}_{E,\langle -,-\rangle})^2 \mid \{X, C^\infty(M)\} \subseteq I^l_p, \{X, \Gamma(E)\} \subseteq I^{l-1}_p \Gamma($$

$$C^\infty(\mathcal{M}_{E,\langle -,-\rangle})^3_{p,l} := \Big\{ X \in C^\infty(\mathcal{M}_{E,\langle -,-\rangle})^3 \mid \{X, C^\infty(M)\} \subseteq I^l_p \Gamma(E),$$
$$\{X, \Gamma(E)\} \subseteq C^\infty(\mathcal{M}_{E,\langle -,-\rangle})^2_{p,l} \Big\},$$

$$C^\infty(\mathcal{M}_{E,\langle -,-\rangle})^4_{p,l} := \Big\{ X \in C^\infty(\mathcal{M}_{E,\langle -,-\rangle})^4 \mid \{X, C^\infty(M)\} \subseteq C^\infty(\mathcal{M}_{E,\langle -,-\rangle})^2_{p,l},$$
$$\{X, \Gamma(E)\} \subseteq C^\infty(\mathcal{M}_{E,\langle -,-\rangle})^3_{p,l-1} \Big\}.$$

Lemma 3.5.30. *Setting*

$$\mathfrak{g}^0_{CA}(p,k) := C^\infty(\mathcal{M}_{E,\langle -,-\rangle})^2_{p,1},$$
$$\mathfrak{g}^1_{CA}(p,k) := C^\infty(\mathcal{M}_{E,\langle -,-\rangle})^3_{p,k},$$
$$\mathfrak{g}^2_{CA}(p,k) := C^\infty(\mathcal{M}_{E,\langle -,-\rangle})^4_{p,2k},$$

these subspaces satisfy

$$\{\mathfrak{g}^0_{CA}(p,k), \mathfrak{g}^i_{CA}(p,k)\} \subseteq \mathfrak{g}^i_{CA}(p,k),$$
$$\{\mathfrak{g}^1_{CA}(p,k), \mathfrak{g}^1_{CA}(p,k)\} \subseteq \mathfrak{g}^2_{CA}(p,k).$$

Now we define:

Definition 3.5.31. Let (E, Θ) be a Courant algebroid, and let $p \in M$. Let $k \geq 0$ be an integer. We say that p is a *fixed point of order k* if $\Theta \in C^\infty(\mathcal{M}_{E,\langle -,-\rangle})^3_{p,k}$.

Remark 3.5.32.

i) In the anchor-bracket description of a Courant algebroid, this means that the anchor vanishes up to order k, and the bracket vanishes up to order $k-1$.

ii) For a Courant algebroid (E, Θ), the algebra $C^\infty(\mathcal{M}_{E,\langle -,-\rangle})$ has an explicit description in [CM21] in terms of multilinear maps $C^\bullet(E)$, which first appeared in [KW15], analogous to the deformation complex of [CM08] for Lie algebroids. The Courant algebroid structure Θ induces a differential on the algebra by $\{\Theta, -\}$, and in terms of the description in [CM21] this differential satisfies a de Rham-type formula.

iii) In terms of the complex given in [CM21], these subspaces can be described as follows. Omitting the argument $\mathcal{M}_{E,\langle -,-\rangle}$, we have for $k = 2, 3, 4$

$$(C^\infty)^k_{p,l} \cong \{\omega \in C^k(E) \mid \omega(e_1, \ldots, e_{k-1}, -) \in I_p^{l-1}\Gamma(E^*),$$
$$\sigma_\omega(e_1, \ldots, e_{k-2}) \in I_p^l \mathfrak{X}(M)\ \forall e_1, \ldots, e_{k-1} \in \Gamma(E)\}.$$

Remark 3.5.33. As before, we have isomorphisms of $\mathfrak{g}^i_{CA}/\mathfrak{g}^i_{CA}(p,k)$ with various jet spaces at p. More precisely, after picking a coordinate neighborhood of p such that $(E, \langle -,-\rangle)$ is trivial as a *pseudo-euclidean* vector bundle, we can make the following identifications.

$$\mathfrak{g}^0_{CA}/\mathfrak{g}^0_{CA}(p,k) \cong T_pM$$

$$\mathfrak{g}^1_{CA}/\mathfrak{g}^1_{CA}(p,k) \cong J_p^{k-1}(E^*[-1] \otimes TM) \oplus J_p^{k-2}(S^3(E^*[-1]))$$

$$\mathfrak{g}^2_{CA}/\mathfrak{g}^2_{CA}(p,k) \cong J_p^{2k-2}(S^2(E^*[-1]) \otimes TM) \oplus J_p^{2k-3}(S^4(E^*[-1])) \oplus J_p^{2k-1}(S^2(TM)).$$

Remark 3.5.34. We can describe the cohomology $H^1(\mathfrak{g}_{CA}/\mathfrak{g}_{CA}(p,1), \overline{\{\Theta, -\}})$ when $p \in M$ is a fixed point (of order 1) of a Courant algebroid Θ on $(E, \langle -,-\rangle)$. Analogous to Lie algebroids, the Courant algebroid structure gives rise to a Lie algebra structure on $E_p := \mathfrak{g}$, and this Lie algebra acts on T_pM by means of a representation with representation map $\tau : \mathfrak{g} \to \mathrm{End}(T_pM)$. Additionally, $\mathfrak{g}$ has an invariant non-degenerate pairing $\langle -,-\rangle_p$, and the stabilizer Lie algebras $\mathfrak{g}_q$ for the representation are coisotropic for all $q \in T_pM$. We can describe the cohomology appearing in Theorem 3.5.36 explicitly in terms of $\mathfrak{g}$, $\langle -,-\rangle_p$, T_pM and τ. As we may assume the bundle E is trivial, with a constant pairing, we can trivialize the algebra of functions $C^\infty(\mathcal{M}_{E,\langle -,-\rangle})$ canonically, and by unpacking the definition of $H^1\left(\mathfrak{g}_{CA}/\mathfrak{g}_{CA}(p,k), \overline{\{\Theta,-\}}\right)$ we obtain the complex given by

$$T_pM \xrightarrow{d^0} \mathfrak{g}^*[-1]\otimes T_pM \xrightarrow{\begin{pmatrix} d_1^1 \\ d_2^1 \end{pmatrix}} S^2(\mathfrak{g}^*[-1])\otimes T_pM \oplus S^2(T_pM)\otimes (T_p^*M\oplus\mathbb{R}).$$

Here we interpret $J_p^1(S^2(TM)) \cong S^2(T_pM)\otimes(T_p^*M\oplus\mathbb{R})$, and view this space as linear maps $T_pM\oplus\mathbb{R}\to S^2(T_pM)$.
Now d^0 and d_1^1 are given by the standard Chevalley-Eilenberg formulas, while d_2^1 is given as follows: Let $\gamma:\mathfrak{g}\to T_pM$ and let $\{e_i\}_{i=1}^r$ be a basis of $\mathfrak{g}$, and $\{e^i\}_{i=1}^r$ the dual basis with respect to $\langle -,-\rangle_p$. Then for $v\in T_pM, \lambda\in\mathbb{R}$, we set

$$d_2^1(\gamma)(v,\lambda) = \sum_{i=1}^r \gamma(e^i)\cdot\tau(e_i)(v) \in S^2(T_pM),$$

where $\cdot$ is the symmetric product.
d_2^1 can be interpreted as follows: First note that $\mathrm{End}(T_pM)\cong T_p^*M\otimes T_pM \cong \mathfrak{X}_{lin}(T_pM)$. In particular, using τ, one can construct the transformation Lie algebroid on the trivial $\mathfrak{g}$-bundle over T_pM. Now if $\gamma:\mathfrak{g}\to T_pM$ is Chevalley-Eilenberg cocycle, then $\tau+\gamma$, which can be interpreted as an affine action, defines a new Lie algebroid structure on $\mathfrak{g}\times T_pM$ analogous to [CF05, Proposition 4.1]. d_2^1 infinitesimally measures the failure of the stabilizers of this Lie algebroid to be coisotropic again, serving as an infinitesimal obstruction to giving rise to a new Courant algebroid structure on $(E,\langle -,-\rangle)$, as shown in [LBM09].

iii) As the spaces are once again jet spaces of vector bundles at points, we can pick the necessary splittings.

iv) Pick a Courant algebroid structure Θ on E such that $p\in M$ is a fixed point of order k.

We check that the data satisfies assumptions 3.3.17a)-e).

a) We pick the following topologies:

  - On $\mathfrak{g}^0$, we pick the C^∞-topology,

- On $\mathfrak{g}^1$, we pick the C^{2k-1}-topology,
- On $\mathfrak{g}^2$, we pick the C^{2k-1}-topology.

b) As $\partial = 0$, it is continuous.

c) The bracket is continuous for the same reason as for Lie algebroids.

d) To understand the gauge action in $\mathfrak{g}$, we take a closer look at $\mathfrak{g}^0$.

Lemma 3.5.35 ([Roy99]).

$$\mathfrak{g}^0 \cong CDO(E, \langle -, - \rangle),$$

the infinitesimal vector bundle automorphisms of E preserving the pairing.

One can show that the action is by vector bundle automorphisms, and the analogue of Lemma 3.3.12 holds in this context. This identifies a neighborhood of $0 \in \mathfrak{g}^0_{CA}/\mathfrak{g}^0_{CA}(p,k)$ with a neighborhood of $p \in M$.

e) Lemma 3.3.19 implies that the gauge action preserves Maurer-Cartan elements.

3.5.2.3 Applying the main theorem

Let $\mathfrak{g}_{CA}$ and $\mathfrak{g}_{CA}(p,k)$ as above. Plugging this into the main theorem yields:

Theorem 3.5.36. *Let (E, Θ) be a Courant algebroid over M. Let $p \in M$ be a fixed point of order k for $k \geq 0$, that is, $\Theta \in \mathfrak{g}^1_{CA}(p,k)$. Assume that*

$$H^1(\mathfrak{g}_{CA}/\mathfrak{g}_{CA}(p,k), \overline{\{\Theta, -\}}) = 0.$$

Then for every open neighborhood U of $p \in M$, there exists a C^{2k-1}-neighborhood $\mathcal{U}$ of $\Theta \in \mathfrak{g}_{CA}$ such that for any Courant algebroid structure $\Theta' \in \mathcal{U}$ there is a family $I \subseteq U$ of fixed points of order k of Θ parametrized by an open neighborhood of

$$0 \in H^0(\mathfrak{g}_{CA}/\mathfrak{g}_{CA}(p,k), \overline{\{\Theta, -\}}).$$

Remark 3.5.37. We compare Theorem 3.5.36 with Theorems 3.4.30 and 3.5.6.

- There are some parallels between Theorems 3.5.6 and 3.5.36, as $T^*[2]A^*[1]$ is a symplectic degree 2 manifold, and the function $\Pi_A + f_{d_{A^*}}$ induces a Courant algebroid structure. However, the notion of fixed points is different: whereas for a Courant algebroid, a point $p \in M$ is fixed of order k if in particular the anchor has vanishing $k-1$-jet at p, for a fixed point $p \in M$ of a Lie algebroid structure on A^* only the anchor of A^* is required to have vanishing $k-1$-jet at p. When $A = TM$ with its standard Lie algebroid structure, this is especially clear: the Courant algebroid structure on $TM \oplus T^*M$ has anchor $\mathrm{id} \oplus \pi^\sharp$, which is surjective, while Theorem 3.5.6 is about fixed points of $\pi^\sharp$.

- Given a degree 2 graded manifold, it is non-canonically isomorphic to a Lie 2-algebroid. In this case all one needs to make this identification is a connection on E which is compatible with the pairing. The multibrackets of the corresponding Lie 2-algebroid can be found in [JL19]. In particular, one could also try to use this identification to get a stability result for Courant algebroids, using only Theorem 3.4.30 if the induced brackets satisfy the assumptions of the theorem. In this case the fixed point p of order k for Θ would be of order (k,k) for the induced Lie 2-algebroid structure Q_Θ. Note that this makes sense, as for $k \geq 2$, we have $k \leq 2k-2$, and for $k=1$, there is no restriction on l. If one is only interested in nearby Courant algebroids however, Theorem 3.5.36 is actually an improvement as the only independent operations in a Courant algebroid are the anchor and the bracket. The unary and ternary bracket are derived from this, which is reflected in the fact that $H^1(\mathfrak{g}_{CA}/\mathfrak{g}_{CA}(p,k), \overline{\{\Theta,-\}})$ only carries information about the anchor and the bracket, while $H^1(\mathfrak{g}_{LnA}/\mathfrak{g}_{LnA}(p,(k,k), \overline{\{\Theta,-\}})$ also carries information about the unary and ternary bracket.
  In terms of graded geometry, this can be interpreted as forgetting the symplectic structure of the underlying graded manifold and considering all homological vector fields, rather than just the symplectic (or Hamiltonian) ones.

3.5.2.4 Examples

Example 3.5.38. According to [LBM09], any quadratic Lie algebra together with a representation on a vector space with coisotropic stabilizer algebras gives rise to a Courant algebroid. In particular, the origin will be a fixed point of order 1.
One way to obtain a quadratic Lie algebra with a representation such that its stabilizer algebras are coisotropic is as follows. Let $\mathfrak{g}$ be any Lie algebra, and V a representation of $\mathfrak{g}$ with representation map $\rho : \mathfrak{g} \to \mathrm{End}(V)$. Let $\mathfrak{g}^*$ be the linear of dual of $\mathfrak{g}$, equipped with the coadjoint representation. Then the semi-direct product $\mathfrak{d} := \mathfrak{g} \ltimes \mathfrak{g}^*$ is a quadratic Lie algebra with respect to the standard pairing, and if we extend ρ to $\mathfrak{d}$ trivially, this yields a representation of $\mathfrak{d}$ with coisotropic stabilizers.

Following remark 3.5.34, the cohomology

$$H^1(\mathfrak{g}_{CA}/\mathfrak{g}_{CA}(p,k), \overline{\{\Theta, -\}}) \tag{3.20}$$

can now be described more explicitly, where Θ is the Courant algebroid structure induced by ρ as in [LBM09]. The coboundaries coincide with the Chevalley-Eilenberg coboundaries for the representation ρ of $\mathfrak{g}$, because $\mathfrak{g}^*$ acts trivially. The same holds for the cocycles coming from $\mathfrak{g}^* \otimes V$. A map $\gamma : \mathfrak{g}^* \to V$ is a cocycle if and only if the following two conditions are satisfied:

- γ intertwines the coadjoint action and ρ,
- For some linear basis $\{e_i\}_{i=1}^n$ of $\mathfrak{g}$ with corresponding dual basis $\{e^i\}_{i=1}^n$ of $\mathfrak{g}^*$, the expression

$$\sum_{i=1}^n \gamma(e^i) \otimes \rho(e_i)(v) \in \wedge^2 V \subseteq V \otimes V \tag{3.21}$$

  for all $v \in V$.

In particular, the cohomology (3.20) vanishes precisely when for any module map $\gamma : \mathfrak{g}^* \to V$, the expression (3.21) has nonzero symmetric part.

Example 3.5.39. Let $\mathfrak{g}$ be a simple Lie algebra, and let $\mathfrak{d} = \mathfrak{g} \ltimes \mathfrak{g}^*$ be the semidirect product with its coadjoint representation as above. Let W be any non-trivial irreducible representation of $\mathfrak{g}$ for which the complexification is an irreducible $\mathfrak{g} \otimes \mathbb{C}$-representation, and let $V = \mathfrak{g}^* \oplus W$. Then we claim that the cohomology as described in the previous example vanishes. By Whitehead's first lemma, we have $H^1_{CE}(\mathfrak{g}, V) = 0$, so in order to prove that the cohomology (3.20) vanishes, we need to show that for any nonzero module map $\gamma : \mathfrak{g}^* \to V$, there exists $v \in V$, such that the expression (3.21) is not skew-symmetric. For this we have to distinguish two cases:

- $W \neq \mathfrak{g}^*$: In this case, by Schur's lemma, any module map $\gamma : \mathfrak{g}^* \to V$ is a multiple of the inclusion. Then for any $v \in W$ which is not invariant under the $\mathfrak{g}$-action, the expression (3.21) is not skew-symmetric.
- $W = \mathfrak{g}^*$: By Schur's lemma, the only module maps are multiples of the inclusions $\iota_i : \mathfrak{g}^* \to V$ for $i = 1, 2$. Let $\lambda, \mu \in \mathbb{R}$. Then for $\gamma = \lambda\iota_1 + \mu\iota_2$, $v = (\phi, \psi) \in \mathfrak{g}^* \oplus \mathfrak{g}^*$ such that $\lambda\phi + \mu\psi \neq 0$. Then (3.21) is not skew-symmetric.

Example 3.5.40. When $W = 0$ as in example 3.5.39, the fixed point is not stable: let $\epsilon > 0$, then modifying the anchor by adding the constant extension of $\epsilon\mathrm{id} : \mathfrak{g}^* \to \mathfrak{g}^*$ yields an affine action of $\mathfrak{g} \ltimes \mathfrak{g}^*$ on $\mathfrak{g}^*$ such that the isotropy Lie algebras at a point $\phi \in \mathfrak{g}^*$ are Lagrangian. By [LBM09] there is a transitive Courant algebroid structure on the trivial $\mathfrak{g} \oplus \mathfrak{g}^*$-bundle over $\mathfrak{g}^*$, which in particular means that it has no fixed points. Consequently, by Theorem 3.5.36, the cohomology is nonzero: Indeed, $\gamma := \mathrm{id} : \mathfrak{g}^* \to \mathfrak{g}^*$ is a nontrivial cocycle.

3.5.3 Fixed points of Dirac structures in a split Courant algebroid

In this section we look at fixed points of Dirac structures in split Courant algebroids. For an introduction to Dirac geometry we refer to [Bur11].
We assume that we are in the following setting: let $((L, d_L), (L^*, d_{L^*}))$ be a Lie bialgebroid over M. In this case, $L \oplus L^*$ can be given a Courant algebroid structure, such that L, L^* are Dirac structures. Assume that $p \in M$ is a fixed point of d_L, that is, if $\rho_L : L \to TM$ denotes the anchor of L, we have $(\rho_L)_p = 0$. When is it the case that for any *Dirac* structure near L, there exists a fixed point $q \in M$ near p? Of course, Theorems 3.2.8 and 3.5.6 could be used for this, as any Dirac structure is in particular a Lie algebroid structure compatible with the Lie algebroid structure d_{L^*} on L^*. Nevertheless, we try to apply the main theorem to this setting: we observe that there is a differential graded Lie algebra $\mathfrak{g}_{Dir}$ such that its Maurer-Cartan elements are precisely Dirac structures near L, and a differential graded Lie subalgebra $\mathfrak{g}_{Dir}(p, 1)$ such that its Maurer-Cartan elements are the Dirac structures near L with a fixed point at p. The conclusion of the main theorem will be of a different form this time: first, the differential in $\mathfrak{g}_{Dir}$ need not be inner, i.e. of the form $[\pi, -]$ for some $\pi \in \Gamma(S^2(L^*[-1]))$. Therefore, we cannot get away with using a graded Lie algebra, which implies that the gauge equation is inhomogeneous. Further, the notion of gauge equivalence does not simply move the fixed point around on the entire manifold M: it only allows to move the fixed point along the leaf of (L^*, d_{L^*}) through p.

3.5.3.1 Dirac structures

We first define Dirac structures in a Courant algebroid $(E, \Theta, \langle -, - \rangle)$ where the pairing has split signature, note that this implies that $\mathrm{rk}E = 2n$ for some $n \geq 0$.

Definition 3.5.41. A *Lagrangian subbundle* $L \subseteq E$ is a rank n subbundle of E such that

$$\langle L, L \rangle \equiv 0.$$

A Lagrangian subbundle $L \subseteq E$ is a *Dirac structure* if

$$[\![\Gamma(L), \Gamma(L)]\!] \subseteq \Gamma(L).$$

In this case the Courant algebroid structure Θ of E restricts to a Lie algebroid structure on L.

Let $(E, \Theta, \langle -, - \rangle)$ be a Courant algebroid, and let L be a Lagrangian subbundle. Assume that there is a Lagrangian subbundle $R \subseteq E$ such that $L \oplus R = E$. Then the pairing

$$\langle -, - \rangle : L \otimes R \to \mathbb{R}$$

is necessarily non-degenerate, identifying $R \cong L^*$. Moreover, under this identification,

$$\langle -, - \rangle : E \otimes E \to \mathbb{R}$$

becomes the standard symmetric pairing on $L \oplus L^*$ given by

$$\langle (x, \alpha), (y, \beta) \rangle = \alpha(y) + \beta(x).$$

It can be shown that such a Lagrangian complement always exists.
Now assume that L is Dirac, and that $R \cong L^*$ is also Dirac. The induced Lie algebroid structures d_L and d_{L^*} on L and L^* respectively now form a Lie bialgebroid $((L, d_L), (L^*, d_{L^*}))$. Moreover, the Courant bracket of E can be recovered from the Lie algebroid structures of L and L^*, see [LWX97].

Remark 3.5.42. The existence of a *Dirac* complement to L in E is non-trivial: it can be shown that in $\mathbb{T}M^H$ (see [LBM09, Example 1.2b]), the generalized tangent bundle of M twisted by a closed 3-form H, T^*M has a Dirac complement precisely when H is exact: as any Lagrangian complement is necessarily the graph of a 2-form ω, it can be shown that the graph is closed under the Courant bracket if and only if $H = d\omega$.
Here, we only discuss the case for when such a Dirac complement exists. The reason for that is that in the general case, the deformations are not governed by a differential graded Lie algebra, but by an L_∞-algebra with a ternary bracket measuring the failure of a Lagrangian complement to be Dirac. In view of remark 3.3.21viii), we will address this in future work.

3.5.3.2 The ingredients

We check that we are in the setting of assumptions 3.3.17.

i) We first identify the algebraic framework behind Dirac structures. As before, we consider the Lie bialgebroid $((L, d_L), (L^*, d_{L^*}))$, with the

induced Courant algebroid structure on $E = L \oplus L^*$. Note that $L \subseteq L \oplus L^*$ is a Dirac structure, which is transverse to L^*. Dirac structures close to L will still be transverse to L^*, hence they can be written as the graph of a bundle map $A : L \to L^*$.
Now because the pairing restricted to the subbundle $\mathrm{gr}(A) := \{(l, A(l)) \in L \oplus L^* \mid l \in L\}$ needs to be identically zero, it follows that A is *skew-symmetric*: it can be interpreted as an element $A \in \Gamma(S^2(L^*[-1])) \subseteq \Gamma(L^*[-1] \otimes L^*[-1])$. The condition that L is Dirac can be written as a Maurer-Cartan equation:

Lemma 3.5.43. *[LWX97] Let $A \in \Gamma(S^2(L^*[-1]))$. Then the graph of $A^\# : L \to L^*$ is Dirac if and only if*

$$d_L(A) + \frac{1}{2}[A, A]_{L^*} = 0.$$

This is the Maurer-Cartan equation in the differential graded Lie algebra

$$(\Gamma(S(L^*[-1]))[1], d_L, [-, -]_{L^*}).$$

Here we use that $((L, d_L), (L^*, d_{L^*}))$ is a Lie bialgebroid.

Definition 3.5.44. Let $((L, d_L), (L^*, d_{L^*}))$ be a Lie bialgebroid. We define

$$\mathfrak{g}_{Dir} := \Gamma(S(L^*[-1]))[1],$$

with differential d_L, and bracket $[-, -]_{L^*}$.

ii) We now identify the graded Lie subalgebra $\mathfrak{g}_{Dir}(p, 1) \subseteq \mathfrak{g}_{Dir}$ such that the graph of a Maurer-Cartan element $\pi \in \mathfrak{g}_{Dir}(p, 1)$ is a Dirac structure for which $p \in M$ is a fixed point.
Assume that p is a fixed point of d_L and let $A \in \Gamma(S^2(L^*[-1]))$. As the anchor of the graph of $A^\#$ is given by

$$\rho_L + \rho_{L^*} \circ A^\# : L \to TM,$$

where we identify $\mathrm{gr}(A^\#) \cong L$, we see that

$$p \in M \text{ is a fixed point for } \mathrm{gr}(A^\#) \iff \mathrm{im}(A^\#_p) \subseteq \ker((\rho_{L^*})_p)$$

$$\iff A_p \in S^2(\ker((\rho_{L^*})_p)[-1]) \subseteq S^2(L^*_p[-1]).$$

This hints at how to pick out subspaces of $\mathfrak{g}_{Dir}$:

Lemma 3.5.45. *Let $i \geq 0$ be an integer. Set*

$$\mathfrak{g}^i_{Dir}(p, 1) := \{\Lambda \in \Gamma(S^{i+1}(L^*[-1])) \mid \Lambda_p \in S^{i+1}(\ker((\rho_{L^*})_p)[-1]) \subseteq S^{i+1}(L^*_p[-1])\}.$$

Then $\mathfrak{g}_{Dir}(p, 1)$ is closed under the Lie bracket $[-, -]_{L^*}$ and differential d_L.

Proof. Note that because both $[-,-]_{L^*}$ and d_L are derivations of the wedge product, it is sufficient to show that for $X, Y \in \mathfrak{g}^0_{Dir}(p,1)$,

$$[X,Y]_{L^*} \in \mathfrak{g}^0_{Dir}(p,1)$$

and

$$d_L(X) \in \mathfrak{g}^1_{Dir}(p,1).$$

The first is easy to show:

$$\rho_{L^*}([X,Y]_{L^*})(p) = [\rho_{L^*}(X), \rho_{L^*}(Y)](p) = 0,$$

as the Lie bracket of two vector fields vanishing at p vanishes at p. Next, in order to prove the second requirement on X, we note that for $\Lambda \in \Gamma(S^2(L^*[-1]))$, we have

$$\Lambda \in \mathfrak{g}^1_{Dir}(p,1) \iff [\Lambda, C^\infty(M)]_{L^*} \subseteq I_p\Gamma(L^*).$$

Now for $f \in C^\infty(M)$

$$[d_L(X), f] = d_L(\rho_{L^*}(X)(f)) - [X, d_L(f)]_{L^*},$$

as $((L, d_L), (L^*, d_{L^*}))$ form a Lie bialgebroid. The first term lies in $I_p\Gamma(L^*)$ because p is a fixed point of d_L, while the second term lies in $I_p\Gamma(L^*)$ because additionally $X \in \mathfrak{g}^0_{Dir}(p,1)$. □

Now the quotients $\mathfrak{g}_{Dir}/\mathfrak{g}_{Dir}(p,1)$ are finite-dimensional vector spaces:

Lemma 3.5.46.

$$\mathfrak{g}^i_{Dir}/\mathfrak{g}^i_{Dir}(p,1) \cong S^{i+1}(L^*_p[-1])/S^{i+1}(\ker((\rho_{L^*})_p)[-1]).$$

Proof. The map

$$\Gamma(S^{i+1}(L^*[-1])) \to S^{i+1}(L^*_p[-1])/S^{i+1}(\ker((\rho_{L^*})_p[-1]))$$

given by evaluating a section at p, and taking the equivalence class mod $S^{i+1}(\ker((\rho_{L^*})_p)[-1])$ is surjective with kernel precisely $\mathfrak{g}^{i+1}_{Dir}(p,1)$. □

iii) By restricting to a small neighborhood of $p \in M$, we may again assume that we have splittings $\sigma_i : \mathfrak{g}^i_{Dir}/\mathfrak{g}^i_{Dir}(p,1) \to \mathfrak{g}^i_{Dir}$.

iv) As $0 \in \Gamma(S^2(L^*[-1]))$ represents the Dirac structure L, we want to look for Maurer-Cartan elements near 0.

We check that the data satisfies assumptions 3.3.17a)-e).

a) We pick the following topologies on $\mathfrak{g}_{Dir}$:

   - We pick the C^∞-topology on $\mathfrak{g}^0_{Dir}$,
   - We pick the C^1-topology on $\mathfrak{g}^1_{Dir}$,
   - We pick the C^0-topology on $\mathfrak{g}^2_{Dir}$.

b) As the value of $d_L(a)$ depends linearly on the 1-jet of $a \in \Gamma(S^2(L^*[-1]))$, d_L is continuous.

c) As the value of $[a,b]_{L^*}$ depends bilinearly on the 1-jets of $a, b \in \Gamma(S^2(L^*[-1]))$, it is continuous.

d) An important difference from all the cases considered so far is the gauge action, which we would like to understand in order to interpret the main theorem. In this example, as d_L is not necessarily an inner derivation of the Lie bracket, we have to solve an *inhomogenous* initial value problem. We first give a general formula for the solution to the gauge equation and then interpret it in this case.
   Let $X \in \Gamma(L^*), \pi \in \Gamma(S^2(L^*[-1]))$. Recall the initial value problem we are interested in:
   $$\frac{d}{dt}\pi_t = [X, \pi_t] - d_L X, \pi_0 = \pi \tag{3.22}$$
   We give the solution in the following lemma, of which the proof is a computation (see Appendix 3.6.2 for the notation):

   Proposition 3.5.47. *Let $D = [X, -]_{L^*} : \Gamma(L^*) \to \Gamma(L^*)$ be the covariant differential operator with symbol $\rho_{L^*}(X)$, and denote by $\tilde{\Phi}^D_t$ its flow until time t (if it exists), as well as its extension to the shifted symmetric powers of L^*. Then*
   $$\pi_t := \tilde{\Phi}^D_{-t}(\pi) - \int_0^t \tilde{\Phi}^D_{-s}(d_L X)\mathrm{d}s \tag{3.23}$$
   satisfies equation (3.22).

   Now we are interested in how the anchor of $\mathrm{gr}(\pi_t)$ changes with t. In particular, if π_1 exists, how the anchor of $\mathrm{gr}(\pi_1)$ compares to the anchor of $\mathrm{gr}(\pi_0)$. We give the answer here, but postpone the proof until the appendix.
   Let $(\tilde{\Phi}^D_t)^* : \Gamma(L) \to \Gamma(L)$ be the dual automorphism. Then:

   Lemma 3.5.48.
   $$\rho_{gr(\pi_t)} = (\phi^X_{-t})_* \circ \rho_{gr(\pi_0)} \circ (\tilde{\Phi}^D_{-t})^*,$$
   where $(\phi^X_{-t})_*$ is the pushforward by the time $-t$-flow of $\rho_{L^*}(X)$.

As $\mathfrak{g}^0_{Dir}/\mathfrak{g}^0_{Dir}(p,1) = L^*_p/\ker((\rho_{L^*})_p) \cong T_pS$, where S is the leaf of L^* going through $p \in M$, and by the Lie algebroid splitting Theorem [Loj02, Theorem 1.1], we see that we can pick splittings such that the gauge action exists for all elements in the image of σ_0. In particular, Lemma 3.5.48 implies that

$$\pi^{\sigma_0(v)} \in \mathfrak{g}^1_{Dir}(p,1) \iff \phi_1^{\sigma_0(v)}(p) \text{ is a fixed point of } \mathrm{gr}(\pi_0),$$

and a neighborhood of $0 \in \mathfrak{g}^0_{Dir}/\mathfrak{g}^0_{Dir}(p,1)$ corresponds to a neighborhood of $p \in S$, using the gauge action.

Remark 3.5.49. Due to Marco Zambon, there is a more geometric argument to interpret the gauge action. When $X \in \Gamma(L^*)$, this induces an element of $\Gamma(L \oplus L^*)$, which has a Courant bracket $[\![-,-]\!]$. X then induces a covariant differential operator $[\![X,-]\!]$ with symbol $\rho_{L^*}(X)$. The time-1 flow of this operator transforms the graph of π_0 into the graph of π_1.

e) Lemma 3.3.19 implies that the gauge action preserves Maurer-Cartan elements.

3.5.3.3 Applying the main theorem

Applying the main theorem to $\mathfrak{g}_{Dir}$ and $\mathfrak{g}_{Dir}(p,1)$ now yields:

Theorem 3.5.50. *Let $p \in M$ be a fixed point of the Dirac structure L inside the Courant algebroid $L \oplus L^*$, that is, $(\rho_L)_p = 0$. Assume that*

$$H^1(\mathfrak{g}_{Dir}/\mathfrak{g}_{Dir}(p,1), \overline{d_L}) = 0.$$

Then for every open neighborhood U of $p \in S$ in S, where S is the (L^*, d_{L^*})-leaf through p, there exists a C^1-neighborhood $\mathcal{U}$ of $0 \in \mathfrak{g}^1_{Dir}$ such that for any Dirac structure $\pi \in \mathcal{U}$ there is a family I in U of fixed points of the Dirac structure $gr(\pi)$ parametrized by an open neighborhood of

$$0 \in H^0(\mathfrak{g}_{Dir}/\mathfrak{g}_{Dir}(p,1), \overline{d_L}).$$

3.5.3.4 Examples

Example 3.5.51 (Poisson structures)**.** The first example we apply this to is the one of Poisson structures: for M a smooth manifold, $\mathbb{T}M = TM \oplus T^*M$ has a Courant algebroid structure, which is just the construction of Section 3.5.1 applied to the standard Lie algebroid structure on TM, and the zero

Lie algebroid structure on T^*M. Now let $\pi \in \Gamma(S^2(TM[-1]))$ be a Poisson structure, that is,

$$[\pi, \pi] = 0.$$

Then the graph of $\pi^\#$ is a Dirac structure, whose fixed points are exactly the zeroes of π. Now let $L = T^*M$, and consider the Dirac structure given by the graph of π. Assume that $p \in M$ is a fixed point of $\mathrm{gr}(\pi)$. Then $d_L : \Gamma(S^\bullet(TM[-1])) \to \Gamma(S^{\bullet+1}(TM[-1]))$ is given by $d_L = [\pi, -]$. Now $\mathfrak{g}_{Dir} = \mathfrak{X}^\bullet(M)$, and $\mathfrak{g}_{Dir}(p, 1) = I_p\mathfrak{X}^\bullet(M)$. Hence the relevant complex is given by

$$T_pM \xrightarrow{\overline{[\pi,-]}} S^2(T_pM[-1]) \xrightarrow{\overline{[\pi,-]}} S^3(T_pM[-1])$$

as in Lemma 3.5.46.
This is precisely the complex appearing in [CF10] for zero-dimensional leaves and [DW06] for $k = 1$.
Finally, the conclusion of the theorems is the same as well. We see that Theorem 3.5.50 recovers the above-mentioned results.

Example 3.5.52 (b-Poisson structures)**.** We now look at a slight variation of this. Let M be a smooth manifold and $Z \subseteq M$ a smooth connected hypersurface. We then consider the Lie algebroid ${}^bTM$, with its standard Lie algebroid structure given by the inclusion $\Gamma({}^bTM) \subseteq \mathfrak{X}(M)$. Now let $\pi \in \Gamma(\wedge^{2\,b}TM)$ be a self-commuting element. This is a Poisson structure for which Z is a Poisson hypersurface, and the graph of $\pi^\# : {}^bT^*M \to {}^bTM$ is a Dirac structure inside ${}^bTM \oplus {}^bT^*M$. Now assume that $p \in M$ is a fixed point of this Dirac structure. One can show that this is is equivalent to $\pi_p = 0 \in \wedge^{2\,b}T_pM$. We apply Theorem 3.5.50 to this example. Note that ${}^bTM$ has two kinds of leaves: there is the hypersurface Z, and the connected components of $M \setminus Z$. As the latter leaves are open and the stability problem is local, $p \in M \setminus Z$ puts us in the ordinary Poisson case. We therefore consider $p \in Z$. In this case $\mathfrak{g}_{Dir} = \Gamma(S({}^bTM[-1]))$, and as $\ker((\rho_{{}^bTM})_p)$ is one-dimensional, we find that the relevant complex is given by

$$T_pZ \xrightarrow{\overline{[\pi,-]}} S^2({}^bT_pM[-1]) \xrightarrow{\overline{[\pi,-]}} S^3({}^bT_pM[-1]) \tag{3.24}$$

as in Lemma 3.5.46.
The conclusion of the theorem tells us now that if the middle cohomology vanishes, we find that for every Poisson structure near π such that Z is a hypersurface, there is a family of zeroes near p inside Z. A natural question is how this compares to the question of fixed points of Poisson structures on Z, starting with the Poisson structure $\pi|_Z$. In this case the relevant complex is given by

$$T_pZ \xrightarrow{\overline{[\pi|_Z,-]}} S^2(T_pZ[-1]) \xrightarrow{\overline{[\pi|_Z,-]}} S^3(T_pZ[-1]). \tag{3.25}$$

There is a surjective chain map from (3.24) to (3.25) which induces a surjection on the middle cohomology. This reflects the fact that any Poisson structure on Z is (locally around $p \in M$) the restriction of a Poisson structure on M for which Z is a Poisson hypersurface.

3.6 Appendix

3.6.1 Auxiliary lemmas

The following lemmas are a small variation of Proposition 4.4 of [CSS14], and are used in the proof of the main theorem of [DW06], as well as Theorem 3.3.20.

Lemma 3.6.1. *Let V, W be finite-dimensional real vector spaces and $B \subseteq W$ a linear subspace of codimension r. Assume that $f : V \to W$ is a smooth map such that $f(0) = 0$, and that $(Df)_0(V) + B = W$. Then for every neighborhood U of $0 \in V$, there exists a C^0-neighborhood $\mathcal{U}$ of f in $C^\infty(V, W)$ such that for every $g \in \mathcal{U}$ there exists $q \in U$ with the property $g(q) \in B$.*
Moreover, there exists a C^1-neighborhood $\mathcal{U}'$ of f such that $g \in \mathcal{U}'$ in addition also satisfies that $(Dg)_q + B = W$.
Finally, in the latter case, the set $g^{-1}(B)$ is a smooth submanifold near $q \in V$ of dimension $\dim_{\mathbb{R}}(\ker(Df)_0)$.

Proof. Let $A \subseteq W$ be a complement to B, and decompose $f : V \to A \oplus B$ as (f_A, f_B). For the first statement, it suffices to show that for every neighborhood U of $0 \in V$, there is a C^0-neighborhood $\mathcal{U}$ of f_A in $C^\infty(V, A)$ such that for every $g \in \mathcal{U}$, there exists $q \in U$ such that $g(q) = 0 \in A$.
Observe that since $f_A : V \to A$ is a submersion at $0 \in V$, up to diffeomorphism we may assume that $V = A \times P$, $U = U_1 \times U_2$ and that f_A is the projection onto A. Picking a basis for A, let $\epsilon > 0$ be small enough such that $[-\epsilon, \epsilon]^r \subseteq U_1$. Let

$$\mathcal{U} = \left\{ g \in C^\infty(V, A) \mid \|f_A - g\|_{[-\epsilon,\epsilon]^r \times \{0_P\},0} < \frac{\epsilon}{2} \right\}$$

be the $\frac{\epsilon}{2}$-ball around f_A with respect to the C^0-seminorm associated to the compact set $K = [-\epsilon, \epsilon]^r \times \{0_P\}$. Now if $g \in \mathcal{U}$, then

$$g|_{A \times \{0_P\}} : \mathbb{R}^r \times \{0_P\} \to \mathbb{R}^r, (x_1, \dots, x_r) \mapsto (g_1(x_1, \dots, x_r), \dots, g_r(x_1, \dots, x_r))$$

satisfies

$$g_i(x_1, \dots, x_{i-1}, -\epsilon, x_{i+1}, \dots x_r) < \frac{\epsilon}{2} + f_i(x_1, \dots, x_{i-1}, -\epsilon, x_{i+1}, \dots x_r) = -\frac{\epsilon}{2} < 0,$$

$$g_i(x_1, \dots, x_{i-1}, \epsilon, x_{i+1}, \dots x_r) > -\frac{\epsilon}{2} + f_i(x_1, \dots, x_{i-1}, \epsilon, x_{i+1}, \dots x_r) = \frac{\epsilon}{2} > 0.$$

By the Poincaré-Miranda theorem, there exists a point $q \in \mathbb{R}^r \cong A$ such that $g(q, 0_P) = 0$.
For the second statement, once we know the existence of q, the derivative of f_A at q is surjective (it is still the projection). As this is an open condition and K is compact, the result follows.
The final statement follows from the preimage theorem, as g intersects B transversely in $q \in V$. □

Remark 3.6.2.

- Note that we can replace B by an open subset of B, by restricting the obtained neighborhood to only those maps which take values in the open subset.

- By using the implicit function theorem instead of the preimage theorem, $\ker(Df)_0$ can be used as a local chart for $g^{-1}(B)$.

Lemma 3.6.3 (Lemma A on p. 61 of [GG80])**.** *Let $f : X \to Y$ be a smooth immersion at a point $p \in X$. Then there is an open neighborhood U of $p \in X$, and a C^1-neighborhood $\mathcal{U}$ of f such that every $g \in \mathcal{U}$ is an injective immersion on U.*

3.6.2 Calculus on vector bundles

Given any vector bundle A over a smooth manifold M, there is a transitive Lie algebroid associated to it. This can be found more generally in the appendix of [CF03]. The sections of this Lie algebroid are given by the space of covariant differential operators:

Definition 3.6.4. Let A be a vector bundle. A *covariant differential operator* on A is an $\mathbb{R}$-linear map

$$D : \Gamma(A) \to \Gamma(A)$$

such that there exists a vector field $\sigma(D) \in \mathfrak{X}(M)$ depending $C^\infty(M)$-linearly on D, satisfying

$$D(fs) - fD(s) = \sigma(D)(f)s$$

for all $f \in C^\infty(M), s \in \Gamma(A)$. The vector field $\sigma(D)$ is called the *symbol of* D. We denote the space of covariant differential operators on A by $CDO(A)$.

We summarize some properties of $CDO(A)$.

Lemma 3.6.5. *Let A be a vector bundle over a smooth manifold M.*

i) For $D \in CDO(A)$, there is a unique $D^* \in CDO(A^*)$ with symbol $\sigma(D)$ such that

$$\sigma(D)(\langle \alpha, s\rangle) = \langle D^*(\alpha), s\rangle + \langle \alpha, D(s)\rangle$$

for $\alpha \in \Gamma(A^*), s \in \Gamma(A)$. Consequently, as $A^{**} \cong A$, $CDO(A) \cong CDO(A^*)$. Here $\langle -, -\rangle$ denotes the dual pairing between A and A^*.

ii) For $D \in CDO(A)$, $k \geq 0$ there is a unique $D^{\otimes k} \in CDO(A^{\otimes k})$ with symbol $\sigma(D)$ such that

$$D(a_1 \otimes \cdots \otimes a_k) = \sum_{i=1}^{k} a_1 \otimes \cdots \otimes D(a_i) \otimes \cdots \otimes a_k$$

for $a_1, \dots, a_k \in \Gamma(A)$. Moreover, the extension $D^{\otimes k}$ descends to all symmetric and exterior powers of A.

A covariant differential operator on a vector bundle A over M can be integrated to a vector bundle automorphism of A. One way to view this is by noting that any $D \in CDO(A)$ with symbol X induces a vector field $\tilde{D}$ on A which preserves the fiberwise constant and fiberwise linear functions, in particular $\tilde{D}$ can be restricted to the zero section $M \subseteq A$, and the induced vector field is X. The flow $\Phi_t^{\tilde{D}}$ is defined for all $t \in \mathbb{R}$ for which the flow ϕ_t^X of X exists. The covariant differential operator can be recovered from the flow by the equation

$$D(s) = \frac{d}{dt}\bigg|_{t=0} \Phi_{-t}^{\tilde{D}} \circ s \circ \phi_t^X$$

for $s \in \Gamma(A)$.
Assume that X is complete. Then for every $t \in \mathbb{R}$, we get an automorphism $\tilde{\Phi}_{-t}^{D}$ of $\Gamma(A)$ defined by

$$\tilde{\Phi}_{-t}^{D}(s) = \Phi_{-t}^{\tilde{D}} \circ s \circ \phi_t^X.$$

This automorphism behaves well with respect to actions described in Lemma 3.6.5:

Lemma 3.6.6. *Let A be a vector bundle over a smooth manifold M, and let $t \in \mathbb{R}$.*

i) For $D \in CDO(A)$, there is a dual automorphism

$$(\tilde{\Phi}_t^D)^* : \Gamma(A^*) \to \Gamma(A^*),$$

determined by the equation

$$\langle (\tilde{\Phi}_t^D)^*(\alpha), \tilde{\Phi}_{-t}^D(s)\rangle = \langle \alpha, s\rangle \circ \phi_t^X$$

for $\alpha \in \Gamma(A^*), s \in \Gamma(A)$*, and* $\langle -,- \rangle$ *denotes the dual pairing between* A *and* A^**. Moreover, the dual automorphism agrees with the integration of the dual covariant differential operator* D^**:*

$$(\tilde{\Phi}_t^D)^* = \tilde{\Phi}_{-t}^{D^*}$$

ii) For $D \in CDO(A), k \geq 0$*, there is an induced automorphism*

$$(\tilde{\Phi}_{-t}^D)^{\otimes k} : \Gamma(A^{\otimes k}) \to \Gamma(A^{\otimes k})$$

determined by the equation

$$(\tilde{\Phi}_{-t}^D)^{\otimes k}(a_1 \otimes \cdots \otimes a_k) = \tilde{\Phi}_{-t}^D(a_1) \otimes \cdots \otimes \tilde{\Phi}_{-t}^D(a_k)$$

for $a_1, \ldots, a_k \in \Gamma(A)$*. Moreover,*

$$(\tilde{\Phi}_{-t}^D)^{\otimes k} = (\tilde{\Phi}_{-t}^{D^{\otimes k}}).$$

Moreover, $(\tilde{\Phi}_{-t}^D)^{\otimes k}$ *descends to all symmetric and exterior powers of* A*.*

Since there is no ambiguity, we will not distinguish between D and its extensions to the exterior, symmetric and tensor powers of A.

3.6.3 Omitted proofs

In this section we prove two statements whose proofs were omitted in the main text. We start with a proof of Lemma 3.4.41, which is about pulling back certain geometric resolutions.

Proof of Lemma 3.4.41. For the first statement, we construct the resolution as follows: pick linear generators $X_1, \ldots, X_r$ of $\mathcal{F}(V)$, and let $\mathcal{F}^{pol} := \langle X_1, \ldots, X_r \rangle_{S(V^*)}$, the $S(V^*)$-submodule of $\mathcal{F}(V)$ consisting of *polynomial* linear combinations of the generators. This is a finitely generated module over the ring of polynomials, so by the Hilbert Syzygy theorem, there exist free modules $F_i = S(V^*) \otimes_{\mathbb{R}} E_i$, where E_i is a finite-dimensional vector space, and module maps $d_i : F_i \to F_{i-1}$ such that

$$0 \longrightarrow F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \ldots \xrightarrow{d_2} F_1 \xrightarrow{\rho} \mathcal{F}^{pol} \longrightarrow 0 \tag{3.26}$$

is exact. Extending this sequence above as sheaf morphisms, we obtain a complex of C_V^∞-modules. As a sequence of sheaves is exact precisely when the sequence is exact on every stalk, we need to show that this implies that for every $q \in V$, the sequence

$$0 \longrightarrow \Gamma_q(E_n) \xrightarrow{d_n} \Gamma_q(E_{n-1}) \xrightarrow{d_{n-1}} \dots \xrightarrow{d_2} \Gamma_q(E_1) \xrightarrow{\rho} \mathcal{F}_q \longrightarrow 0$$

is exact, where $\Gamma_q(E)$ is the stalk of the sheaf of sections of E at q. Note that

$$\Gamma_q(E_i) = C^\infty_{V,q} \otimes_{S(V^*)} F_i = C^\infty_{V,q} \otimes_{C^\omega_{V,q}} C^\omega_{V,q} \otimes_{S(V^*)} F_i,$$

where C^ω_V is the sheaf of analytic functions on V, and $C^\omega_{V,q}$ is an $S(V^*)$-module by means of the inclusion of polynomials as analytic functions. As

$$\mathcal{F}_q = C^\infty_{V,q} \otimes_{S(V^*)} \mathcal{F}^{pol},$$

it suffices to show that $C^\infty_{V,q}$ is flat over $S(V^*)$. We do this in two steps: we show that germs of analytic functions in a point q are flat over polynomials, and that the germs of smooth functions in a point q are flat over germs of analytic functions in q.
The second step is just [Mal66, Corollary VI.1.12]. For the first step, note that we have a commutative triangle,

$$\begin{array}{ccc} S(V^*) & \xrightarrow{T_q} & \hat{S}(V^*) \\ \downarrow & \nearrow_{T_q} & \\ C^\omega_{V,q} & & \end{array}$$

where $\hat{S}(V^*)$ is the algebra of formal power series on V, T_q takes the Taylor expansion around q, and the vertical map is the inclusion. Now we note that T_q is actually the I_q-adic completion map with respect to the vanishing ideal of $q \in V$, which for $C^\omega_{V,q}$ is faithfully flat by [Sta21a]. As the ring of polynomials is not local, we cannot apply this argument to the horizontal map. However, we can use [Sta21b] to conclude that the horizontal map is flat. It then follows from the definition of (faithful) flatness that the vertical map is also flat.
For the second statement, we first find a resolution for $C^\infty_{V\times U} \otimes_{C^\infty_V} \mathcal{F}$. By the assumption that $\mathcal{F}_U$ is an isomodule deformation, this gives a resolution of $\mathcal{F}_U$. The straightforward thing to do here would be to take the resolution we found earlier, and pull it back to $V \times U$. As far as we know, it is not clear whether this preserves exactness. We therefore take a different way. Consider again the resolution (3.26). If we apply the functor $S(V^* \oplus \mathbb{R}^n) \otimes_{S(V^*)} -$, the sequence stays exact. If we now apply the same argument as before, we obtain a geometric resolution of $C^\infty_{V\times U} \otimes_{C^\infty_V} \mathcal{F}$ as in the formulation of the lemma. As the differentials are unchanged, the first two properties are automatic. Finally, the fact that the Lie n-algebroid restricts is a consequence of the fact that $\mathcal{F}_U$ is tangent to $V \times \{0\}$. Now the existence of the Lie n-algebroid structure is guaranteed by [LGLS20, Theorem 2.7]. □

We now prove Lemma 3.5.48 about the anchor of a gauge transformed Dirac structure.

Proof of Lemma 3.5.48. The non-trivial part of the proof is evaluating the integral of equation (3.23). For that we give the antiderivative of $\rho_{L^*} \circ \tilde{\Phi}^D_{-s}(d_L X)^\#$:

Claim.

$$\frac{d}{ds}(\phi^{\rho_{L^*}(X)}_{-s})_* \circ \rho_L \circ (\tilde{\Phi}^D_{-s})^* = -\rho_{L^*} \circ \tilde{\Phi}^D_{-s}(d_L X)^\#.$$

Proof of claim. It suffices to prove that the claim holds when evaluating both sides on a section $a \in \Gamma(L)$.

$$\begin{aligned}
\frac{d}{ds}(\phi^{\rho_{L^*}(X)}_{-s})_* \circ \rho_L \circ (\tilde{\Phi}^D_{-s})^*(a) =& [\rho_{L^*}(X), (\phi^{\rho_{L^*}(X)}_{-s})_* \circ \rho_L \circ (\tilde{\Phi}^D_{-s})^*(a)] \\
& - (\phi^{\rho_{L^*}(X)}_{-s})_* \circ \rho_L \circ \mathcal{L}_X((\tilde{\Phi}^D_{-s})^*(a)) \\
=& (\phi^{\rho_{L^*}(X)}_{-s})_*[\rho_{L^*}(X), \rho_L((\tilde{\Phi}^D_{-s})^*(a))] \\
& - (\phi^{\rho_{L^*}(X)}_{-s})_*(\rho_L(\mathcal{L}_X((\tilde{\Phi}^D_{-s})^*(a)))) \\
\overset{\triangle}{=}& (\phi^{\rho_{L^*}(X)}_{-s})_*(\rho_L(\mathcal{L}_X(\tilde{\Phi}^D_{-s})^*(a))) \\
& - (\phi^{\rho_{L^*}(X)}_{-s})_*(\rho_{L^*}(\mathcal{L}_{(\tilde{\Phi}^D_{-s})^*(a)}(X))) \\
& + (\phi^{\rho_{L^*}(X)}_{-s})_*(\rho_{L^*}(d_L(\iota_{(\tilde{\Phi}^D_{-s})^*(a)}(X))) \\
& - (\phi^{\rho_{L^*}(X)}_{-s})_*(\rho_L(\mathcal{L}_X((\tilde{\Phi}^D_{-s})^*(a)))) \\
=& - (\phi^{\rho_{L^*}(X)}_{-s})_*(\rho_{L^*}(\iota_{(\tilde{\Phi}^D_{-s})^*(a)}(d_L X))) \\
\overset{\star}{=}& - \rho_{L^*}(\tilde{\Phi}^D_{-s}(\iota_{(\tilde{\Phi}^D_{-s})^*(a)}(d_L X))) \\
=& - \rho_{L^*}(\tilde{\Phi}^D_{-s}(d_L X)^\#(a)).
\end{aligned}$$

Here $\mathcal{L}_X = d_{L^*}\iota_X + \iota_X d_{L^*}$, and an analogous formula holds for $\mathcal{L}_{(\tilde{\Phi}^D_{-s})^*(a)}$. Further, at $\triangle$, we apply [LWX97, Lemma 4.3], and at $\star$ we apply the equality

$$\rho_{L^*} \circ \tilde{\Phi}^D_{-s} = (\phi^{\rho_{L^*}(X)}_{-s})_* \circ \rho_{L^*},$$

which holds as both sides of the equation satisfy the ODE

$$\frac{d}{ds}\gamma_s(Y) = [\rho_{L^*}(X), \gamma_s(Y)]$$

for all $Y \in \Gamma(L^*)$, with $\gamma_0 = \rho_{L^*}$. □

To finish the proof, recall from 3.5.47 that

$$\pi_1 = \tilde{\Phi}^D_{-1}(\pi_0) - \int_0^1 \tilde{\Phi}^D_{-s}(d_L X)\mathrm{d}s. \tag{3.27}$$

Then

$$\begin{aligned} \rho_{\mathrm{gr}(\pi_1)} &= \rho_L + \rho_{L^*} \circ \pi_1^\# \\ &\overset{\star}{=} \rho_L + \rho_{L^*} \circ \tilde{\Phi}^D_{-1}(\pi_0)^\# + (\phi^{\rho_{L^*}(X)}_{-1})_* \circ \rho_L \circ (\tilde{\Phi}^D_{-1})^* - \rho_L \\ &= \rho_{L^*} \circ \tilde{\Phi}^D_{-1} \circ \pi_0^\# \circ (\tilde{\Phi}^D_{-1})^* + (\phi^{\rho_{L^*}(X)}_{-1})_* \circ \rho_L \circ (\tilde{\Phi}^D_{-1})^* \\ &= (\phi^{\rho_{L^*}(X)}_{-1})_* \circ (\rho_{L^*} \circ \pi_0^\# + \rho_L) \circ (\tilde{\Phi}^D_{-1})^* \\ &= (\phi^{\rho_{L^*}(X)}_{-1})_* \circ \rho_{\mathrm{gr}(\pi_0)} \circ (\tilde{\Phi}^D_{-1})^*, \end{aligned}$$

where $\star$ uses the claim and (3.27). □

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Chapter 4

Stability of fixed points of Dirac structures

This chapter contains the article [SZ23].

Abstract - Given an L_∞-algebra V and an L_∞-subalgebra W, we give sufficient conditions for all small Maurer-Cartan elements of V to be equivalent to Maurer-Cartan elements lying in W. As an application, we obtain a stability criterion for fixed points of a Dirac structure (for instance a twisted Poisson structure), i.e. points where the corresponding leaf is zero-dimensional. The criterion guarantees that any nearby Dirac structure also has a fixed point.

4.1 Introduction

Stability questions appear naturally in mathematics. For instance, given a vector field X vanishing at a point p, one can ask about the stability of p: does every vector field sufficently close to X have a zero nearby p? Given a Lie algebra structure on a fixed vector space and a Lie subalgebra $\mathfrak{h}$, one can ask about the stability of $\mathfrak{h}$: does every sufficently close Lie algebra structure admit a Lie subalgebra nearby $\mathfrak{h}$?

The main contribution of this paper is two-fold. First we state an algebraic theorem about $L_\infty[1]$-algebras. This theorem can be applied to a variety of stability questions. In the second part of the paper we apply it to a specific geometric problem, obtaining a stability criterion for fixed points of Dirac structures. This includes twisted Poisson structures as a special case, and extends some results obtained in [CF10][DW06][Sin22].

Recall that $L_\infty[1]$-*algebras* are a notion equivalent to the L_∞-algebras introduced by Lada and Stasheff in the 1990's [LS93], in order to provide a "up to homotopy" version of Lie algebras. They contain special elements – called Maurer-Cartan elements – which come equipped with an equivalence relation. Deformation problems are typically governed by such algebraic structures, in the sense that (equivalence classes of) deformations are parametrized by (equivalence classes of) Maurer-Cartan elements of the $L_\infty[1]$-algebra.

We paraphrase our main algebraic results as follows, omitting technical assumptions, and refer to Theorem 4.3.1 for the full statement:

Theorem. *Let V be an $L_\infty[1]$-algebra, whose underlying cochain complex we denote by by (V, d). Let W be an $L_\infty[1]$-subalgebra of finite codimension, and fix a Maurer-Cartan element Q of W. Denote by d^Q is the differential on V obtained twisting d by Q, and view d^Q as a differential on the quotient V/W. If*

$$H^0(V/W, d^Q) = 0,$$

then, under some technical conditions, any Maurer-Cartan element of V sufficently close to Q is equivalent to a Maurer-Cartan element lying in W.

This result extends a previous one on differential graded Lie algebras by the first author [Sin22, Theorem 3.20] (see also the works of Dufour-Wade [DW06] and Crainic-Fernandes [CF05][CF10]).

As an application of the above algebraic theorem, we consider *Dirac structures* [Cou90], geometric structures which include Poisson bivector fields and closed 2-forms, and which can be used to characterize Hitchin's generalized complex structures [Hit03]. Dirac structures are defined as Lagrangian and involutive subbundles of *Courant algebroids* [LWX97]. Important examples of the latter are $TM \oplus T^*M$, endowed with a bracket that depends on a choice of closed 3-form H on the manifold M. In that case, one speaks of H-twisted Dirac structures [ŠW01] (this includes H-twisted Poisson structures). They first appeared in the context of σ-models in physics, in the work of Klimčik-Strobl [KS02] and Park [Par], and in that context H is called the Wess-Zumino-Witten 3-form.

A byproduct of this note is a geometric characterization of the equivalences of Dirac structures induced by the $L_\infty[1]$-algebra governing deformations of Dirac structures: they are given by applying inner automorphisms of the ambient Courant algebroid, see Proposition 4.5.2.

Applying the above algebraic theorem, and upon making explicit the assumptions and conclusions, one obtains Theorem 4.6.7. This is a statement on the stability of fixed points (i.e. zero-dimensional leaves) of Dirac structures . We state a simplified version as follows:

Theorem. *Let $E \to M$ be a Courant algebroid whose pairing has split signature, denote by $\rho\colon E \to TM$ its anchor map. Let $A \subseteq E$ be a Dirac structure which has a fixed point at $p \in M$, i.e. $\rho(A_p) = 0$. Denote by $\mathfrak{g}$ the Lie algebra A_p, and consider the Lie ideal $\mathfrak{h} := (\ker(\rho|_{E_p}))^{\perp}$. Assume that*

$$H^2\left(\frac{\wedge^\bullet \mathfrak{g}^*}{\wedge^\bullet \mathfrak{h}^\circ}, \overline{d_{\mathfrak{g}}}\right) = 0.$$

Then any Dirac structure sufficiently close to A admits a fixed point nearby p, and lying in the leaf of E through p.

This theorem clarifies and improves [Sin22, Theorem 5.50], since it holds in wider generality and without making any auxiliary choice.

We conclude this note presenting some examples of the above theorem in §4.7. For instance, on a Lie group G with a bi-invariant metric we consider the Cartan-Dirac structure, which is twisted by the Cartan-Dirac 3-form H. The identity $e \in G$ is a fixed point of the Cartan-Dirac structure. If the second Lie algebra cohomology vanishes, then any sufficently close H-twisted Dirac structure also has a fixed point nearby e.

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4.2 Background on $L_\infty[1]$-algebras

In this section we recall basic notions about $L_\infty[1]$-algebras. The latter are central objects in deformation theory, and are completely equivalent to L_∞-algebras [LS93], which have ordinary Lie algebras and differential graded Lie algebras as special cases. All vector spaces are assumed to be over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

Definition 4.2.1. An $L_\infty[1]$-*algebra* is a pair $(V, \{\mu_k\}_{k\geq 1})$, where

i) $V = \bigoplus_{i\in\mathbb{Z}} V^i$ is a $\mathbb{Z}$-graded vector space,

ii) for every $k \geq 1$,

$$\mu_k \colon S^k(V) \to V$$

is a multilinear degree 1 map called a multibracket,

satisfying for $n \geq 0$, $x_0, \dots, x_n \in V$,

$$\sum_{i=0}^{n} \sum_{\sigma \in Sh(i+1,n-i-1)} \epsilon(\sigma)\mu_{n-i+1}(\mu_{i+1}(x_{\sigma(0)}, \dots, x_{\sigma(i)}), x_{\sigma(i+1)}, \dots, x_{\sigma(n)}) = 0. \tag{4.1}$$

Here $S^k(V)$ denotes the k-th graded symmetric power of V, and $\epsilon(\sigma)$ is the *Koszul* sign, determined by

$$x_1 \dots x_n = \epsilon(\sigma) x_{\sigma(1)} \dots x_{\sigma(n)}$$

in the graded symmetric algebra $S(V)$.

Remark 4.2.2. The equations (4.1) are higher analogues of the Jacobi identity for Lie algebras.

i) For $n = 0$, it follows that

$$\mu_1^2 = 0,$$

turning (V, μ_1) into a cochain complex.

ii) For $n = 1$, it follows that μ_1 is a graded derivation of μ_2.

iii) For $n = 2$, it follows that μ_3 is a contracting homotopy of the Jacobiator of μ_2, with respect to the differential μ_1.

Now let $(V, \{\mu_k\}_{1 \leq k \leq n})$ be an $L_\infty[1]$-algebra with $\mu_k \equiv 0$ for $k > n$ (i.e., only finitely many multibrackets are non-zero).

Definition 4.2.3. A degree 0 element $Q \in V^0$ is a *Maurer-Cartan element* if

$$\sum_{i=1}^{n} \frac{1}{i!} \mu_i(Q, \dots, Q) = 0. \tag{4.2}$$

A motivation for this definition is the following. For any degree 0 element Q, we can define new structure maps

$$\mu_k^Q := \sum_{i=0}^{\infty} \frac{1}{i!} \mu_{k+i}(\underbrace{Q, \dots, Q}_{i \text{ times}}, -, \dots, -), \tag{4.3}$$

where we note that the sum is finite. A natural question to ask is when these maps define a new $L_\infty[1]$-algebra structure on V. It turns out that if Q is a Maurer-Cartan element, then $(V, \{\mu_k^Q\}_{1\leq k\leq n})$ is an $L_\infty[1]$-algebra (see e.g. the text following [Fukdf, Lemma 2.2.1], or [Dol, §2]).

Given a degree 0 element $Q \in V^0$ and a $X \in V^{-1}$, we can construct a new degree 0 element, denoted by Q^X, as follows. For the following definition, we need V^0 to carry a topology.

Definition 4.2.4. Let $(V, \{\mu_k\}_{1\leq k\leq n})$ be an $L_\infty[1]$-algebra such that for every $i \in \mathbb{Z}$, V^i carries a locally convex topology[1]. For $Q \in V^0$ and $X \in V^{-1}$, assume that the initial value problem

$$\frac{d}{dt} Q_t = \mu_1^{Q_t}(X), \qquad Q_0 = Q \tag{4.4}$$

has a unique solution for all $t \in [0, 1]$ (notice that the right hand side was defined in equation (4.3)). Then we define

$$Q^X := Q_1 \in V^0,$$

the value of the solution at $t = 1$.

Remark 4.2.5. When solving equation (4.4) in terms of formal paths, or when the $L_\infty[1]$-algebra is nilpotent, it can be shown that Q^X is a Maurer-Cartan element if and only if Q is (see for instance [DP16, Corollary 1]). Since we are dealing with differentiable paths, we will need to assume that the solution of (4.4) takes value in the space of Maurer-Cartan elements.

Finally, we will need subspaces of $L_\infty[1]$-algebras which have an induced $L_\infty[1]$-algebra structure.

Definition 4.2.6. Let $(V, \{\mu_k\}_{k\geq 1})$ be an $L_\infty[1]$-algebra, and let $W \subseteq V$ be a graded linear subspace. Then W is said to be a $L_\infty[1]$-*subalgebra* if

$$\mu_k(S^k(W)) \subseteq W$$

for all $k \geq 1$.

4.3 Main theorem for $L_\infty[1]$-algebras

In this subsection we present a general statement about L_∞-algebras and their Maurer-Cartan elements. It generalizes [Sin22, §3.3] from differential graded Lie algebras to L_∞-algebras.

[1] See [Rud91] for some background on locally convex vector spaces, and see [CP82] for some background on function spaces.

Assume that we have the following data:

i) An $L_\infty[1]$-algebra $(V, \{\mu_k\}_{1\leq k\leq n})$ with finitely many non-trivial multibrackets, such that for each $i = -1, 0, 1$, V^i carries a locally convex topology,

ii) a $L_\infty[1]$-subalgebra W of V such that, for $i = -1, 0, 1$, the subspace W^i is of finite codimension and closed in V^i,

iii) linear splittings $\sigma_i : V^i/W^i \to V^i$ for $i = -1, 0$,

iv) a Maurer-Cartan element $Q \in W^0$,

such that

a) the multibrackets $\mu_k : S^k(V^0) \to V^1$ are continuous, when viewed as symmetric $\mathbb{K}$-multilinear maps,

b) there is a convex open neighborhood U of $0 \in V^{-1}/W^{-1}$ such that for every $X \in U$ the following holds: the element $Q^{\sigma_{-1}(X)}$ as in definition 4.2.4 is defined, the assignment

$$U \times V^0 \to V^0, \quad (X, Q') \mapsto (Q')^{\sigma_{-1}(X)}$$

is jointly continuous, and the mod W^0 class of $(Q')^{\sigma_{-1}(X)}$ depends smoothly on $X \in U$ for each fixed Q',

c) for $X \in U$, an element $Q' \in V^0$ is Maurer-Cartan if and only if $(Q')^{\sigma_{-1}(X)}$ is Maurer-Cartan.

Recall that μ_1^Q was defined in eq. (4.3); we denote by $\overline{\mu_1^Q}$ the induced differential on V/W.

Theorem 4.3.1. *Assume that we are in the setting described above. Assume that*

$$H^0(V/W, \overline{\mu_1^Q}) = 0.$$

Then there exists an open neighborhood $\mathcal{U} \subseteq V^0$ of Q such that for any Maurer-Cartan element $Q' \in \mathcal{U}$, there exists a family $I \subseteq U$, smoothly parametrized by an open neighborhood of

$$0 \in \ker(\overline{\mu_1^Q} : V^{-1}/W^{-1} \to V^0/W^0),$$

with the property that $x \in I \implies (Q')^{\sigma_{-1}(x)} \in W^0$.

In particular, Q' is related (in the sense of Definition 4.2.4) to a Maurer-Cartan element lying in W.

Remark 4.3.2. We provide a heuristic interpretation of the theorem. By the conclusions of the theorem, the map of moduli spaces of Maurer-Cartan elements induced by the inclusion

$$MC(W)/\sim \;\longrightarrow\; MC(V)/\sim$$

is surjective nearby $[Q]$. The corresponding map of formal tangent spaces at $[Q]$ is

$$H^0(W, \mu_1^Q) \longrightarrow H^0(V, \mu_1^Q). \tag{4.5}$$

By the hypotheses of the theorem, this linear map is surjective: indeed the obvious short exact sequence of cochain complexes gives rise to a long exact sequence in cohomology, a piece of which reads

$$\cdots \longrightarrow H^0(W, \mu_1^Q) \longrightarrow H^0(V, \mu_1^Q) \to H^0(V/W, \overline{\mu_1^Q}) \longrightarrow \dots$$

While the moduli spaces of Maurer-Cartan elements are not smooth manifolds, and hence the regular value theorem can not be applied, Theorem 4.3.1 shows that the vanishing of $H^0(V/W, \overline{\mu_1^Q})$ is sufficient to obtain the same conclusion. It would be interesting to investigate under what conditions surjectivity of (4.5) implies the conclusions of the theorem. This observation is analogous to the one made at the end of [CSS14, Remark 5.13].

Before proving Theorem 4.3.1, we present the main idea of the proof. The conclusion of the theorem suggests to consider, for every Q' nearby Q, the map

$$\mathrm{ev}_{Q'} : V^{-1}/W^{-1} \to V^0/W^0,\ v \mapsto (Q')^{\sigma_{-1}(v)} + W^0.$$

If the map ev_Q was a submersion in a neighborhood of $0 \in V^{-1}/W^{-1}$, the same would hold for its perturbation $\mathrm{ev}_{Q'}$, implying that its image would contain the origin, as desired. While ev_Q is almost never a submersion, it is transverse to a certain subspace $K \subseteq V^0/W^0$, therefore $\mathrm{ev}_{Q'}$ too; from this, using the cohomological assumption and the Maurer-Cartan condition on Q', we will be able obtain the desired conclusion.

Proof. For simplicity, we take $U = V^{-1}/W^{-1}$, but the proof goes through for any convex open neighborhood of the origin in V^{-1}/W^{-1}. In the first part of the proof we assume the existence of certain maps between the spaces V^i/W^i for $i = -1, 0, 1$ with prescribed properties, from which the result follows. In the second part we explicitly construct the maps.

Assume the existence of the following maps:

1) A smooth map
$$\mathrm{ev}_{Q'} : V^{-1}/W^{-1} \to V^0/W^0$$
depending continuously on $Q' \in V^0$,

2) a smooth map
$$R_{v,Q'} : V^0/W^0 \to V^1/W^1$$
depending continuously on $(v, Q') \in V^{-1}/W^{-1} \times V^0$,

with the following properties:

A) $\mathrm{ev}_Q(0) = 0 \in V^0/W^0$, and the derivative satisfies
$$(D(\mathrm{ev}_Q))_0 = \overline{\mu_1^Q} : V^{-1}/W^{-1} \to V^0/W^0.$$
Moreover, the element $(Q')^{\sigma_{-1}(v)}$ lies in the subspace $W^0 \subseteq V^0$ if and only if $\mathrm{ev}_{Q'}(v) = 0$.

B) $R_{v,Q'}(0) = 0 \in V^1/W^1$ for every $(v, Q') \in V^{-1}/W^{-1} \times V^0$, and the derivative of $R_{0,Q}$ satisfies
$$(D(R_{0,Q}))_0 = \overline{\mu_1^Q} \colon V^0/W^0 \to V^1/W^1.$$

C) Whenever $Q' \in V^0$ is Maurer-Cartan, for every $v \in V^{-1}/W^{-1}$ we have:
$$R_{v,Q'}(\mathrm{ev}_{Q'}(v)) = 0.$$

The following diagram summarizes diagrammatically the above maps.
$$V^{-1}/W^{-1} \xrightarrow{\mathrm{ev}_{Q'}} V^0/W^0 \xrightarrow{R_{v,Q'}} V^1/W^1 \ .$$

The conclusion of the theorem follows exactly as in [Sin22, Theorem 3.20]. We summarize the main ideas for the reader's convenience.

- Let K be a complement to $\ker(\overline{\mu_1^Q})$ in V^0/W^0. Property B) implies that $R_{0,Q}$ restricted to K is an immersion at $0 \in K$. By continuity, for (v, Q') close enough to $(0, Q)$, the same is true for $R_{v,Q'}$. Therefore $R_{v,Q'}$ is injective in a neighborhood O of $0 \in K$. The neighborhood O can be chosen independently of (v, Q').
- Property A) and the cohomological assumption imply that ev_Q intersects K transversely in 0. Therefore, for any Q' close enough to Q, the map $\mathrm{ev}_{Q'}$ also intersects K transversely, and there exists a $v \in V^{-1}/W^{-1}$ close to 0 such that $\mathrm{ev}_{Q'}(v) \in O$.

- When Q' is Maurer-Cartan, using property C), the fact that $R_{v,Q'}(0) = 0$ by property B), and the injectivity in the first item above, it follows that $\mathrm{ev}_{Q'}(v) = 0$. By construction, this means that $(Q')^{\sigma_{-1}(v)} \in W^0$, as desired. More is true: as $\mathrm{ev}_{Q'}^{-1}(\{0\}) = \mathrm{ev}_{Q'}^{-1}(O)$ is non-empty, the transversality argument above implies that $\mathrm{ev}_{Q'}^{-1}(\{0\})$ is a submanifold of dimension equal to the one of $\ker(\overline{\mu_1^Q} : V^{-1}/W^{-1} \to V^0/W^0)$.

We now define the maps $\mathrm{ev}_{Q'}$ and $R_{v,Q'}$ used above.

1) Let $Q' \in V^0$. Then for $v \in V^{-1}/W^{-1}$, we set

$$\mathrm{ev}_{Q'}(v) = (Q')^{\sigma_{-1}(v)} + W^0.$$

Then by condition a) of the data at the beginning of this section, the map depends continuously on Q' and smoothly on $v \in V$.

2) Let $(v, Q') \in V^{-1}/W^{-1} \times V^0$. To shorten the notation we write $X := (Q')^{\sigma_{-1}(v)}$, and $\overline{X} := X + W^0$. For $\overline{Y} \in V^0/W^0$, we set

$$R_{v,Q'}(\overline{Y}) = \sum_{i=1}^{n} \frac{1}{i!}\mu_i\Big((X - \sigma_0(\overline{X})) + \sigma_0(\overline{Y}), \tag{4.6}$$

$$\ldots, (X - \sigma_0(\overline{X})) + \sigma_0(\overline{Y})\Big) + W^1. \tag{4.7}$$

Since the multibrackets μ_i are continuous, the map R depends continuously on the parameters $(v, Q') \in V^{-1}/W^{-1} \times V^0$.

Notice that, as the μ_i are symmetric when the arguments have degree 0, we can use Newton's binomial formula to rewrite (4.6) as

$$R_{v,Q'}(\overline{Y}) = \sum_{i=1}^{n}\sum_{j=0}^{i} \frac{1}{i!}\binom{i}{j}\mu_i(\underbrace{X - \sigma_0(\overline{X}), \ldots, X - \sigma_0(\overline{X})}_{i-j \text{ times}}, \underbrace{\sigma_0(\overline{Y}), \ldots, \sigma_0(\overline{Y})}_{j \text{ times}}) + W^1. \tag{4.8}$$

We check that the above maps satisfy properties A), B), C) above.

A) The property $\mathrm{ev}_Q(0) = 0$ holds since $Q \in W^0$, and the one regarding the value of $\mathrm{ev}_{Q'}$ holds by definition. For the derivative, we compute for

$v \in V^{-1}/W^{-1}$

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} \mathrm{ev}_Q(tv) &= \frac{d}{dt}\Big|_{t=0} (Q)^{t\sigma_{-1}(v)} + W^0 \\ &= \mu_1^{Q^{t\sigma_{-1}(v)}}(\sigma_{-1}(v))\Big|_{t=0} + W^0 \\ &= \overline{\mu_1^Q}(v).\end{aligned}$$

Here in the second equality we used that $(Q)^{t\sigma_{-1}(v)} = \widetilde{Q}_t$, where the latter is the solution of $\frac{d}{dt}\widetilde{Q}_t = \mu_1^{\widetilde{Q}_t}(\sigma_{-1}(v))$ with initial condition $\widetilde{Q}_0 = Q$ (this is a consequence of the fact that the r.h.s. of (4.4) depends linearly on X and the uniqueness of the solution of (4.4).)

B) The property $R_{v,Q'}(0) = 0 \in V^1/W^1$ holds since $X - \sigma_0(\overline{X}) \in W^0$. To compute $(D(R_{0,Q}))_0 = \overline{\mu_1^Q}$ we notice that only the $j = 1$ summand in (4.8) contributes.

C) Finally, let $Q' \in V^0$, $v \in V^{-1}/W^{-1}$. As above we write $X = (Q')^{\sigma_{-1}(v)}$ and $\overline{X} = X + W^0 = \mathrm{ev}_{Q'}(v)$. Then

$$R_{v,Q'}(\mathrm{ev}_{Q'}(v)) = R_{v,Q'}(\overline{X}) = \sum_{i=1}^{n} \frac{1}{i!}\mu_i(X, \dots, X) + W^1.$$

If $Q' \in V^0$ is Maurer-Cartan, then X also is (by condition c) at the beginning of this section), so the above expression vanishes.

□

Remark 4.3.3. The continuity of the multibrackets $\mu_k : S^k(V^0) \to V^1$, required in property a) at the beginning of this section, was used to ensure that the map R defined by

$$\begin{aligned} R : V^{-1}/W^{-1} \times V^0 &\to C^\infty(V^0/W^0, V^1/W^1) \\ (v, Q) &\mapsto R_{v,Q} \end{aligned} \tag{4.9}$$

is continuous see item 2) of the proof of Theorem 4.3.1). Here the right hand side is equipped with the C^1-topology. There is however a different condition to ensure this which is easier to check, and which we provide in Lemma 4.3.4 below.

Lemma 4.3.4. *Assume that there exists a closed subspace $F \subseteq V^0$ of finite codimension such that the multibrackets*

$$\mu_k : S^k(V^0) \to V^1/W^1$$

factor through $S^k(V^0/F)$. Then the map R as in (4.9) *is continuous.*

Proof. The proof is analogous to the proof of [Sin22, Lemma 3.22], with the exception that the map R now takes values in the finite-dimensional subspace consisting of *polynomial* maps $V^0/F \to V^1/W^1$ of degree at most n. $\square$

4.4 Background on Dirac structures and their deformations

We recall the definition of Courant algebroids [LWX97], Dirac structures [Cou90], and following [FZ15] we review an $L_\infty[1]$-algebra governing their deformations.

4.4.1 Courant algebroids

We first need to introduce Courant algebroids [LWX97].

Definition 4.4.1. A *Courant algebroid* over a manifold M is a vector bundle $E \to M$ equipped with a fibrewise non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$, an $\mathbb{R}$-bilinear bracket $[\![\cdot, \cdot]\!]$ on the smooth sections $\Gamma(E)$, and a bundle map $\rho : E \to TM$ called the *anchor*, which satisfy the following conditions for all $e_1, e_2, e_3 \in \Gamma(E)$ and $f \in C^\infty(M)$:

C1) $[\![e_1, [\![e_2, e_3]\!]]\!] = [\![[\![e_1, e_2]\!], e_3]\!] + [\![e_2, [\![e_1, e_3]\!]]\!]$,

C2) $\rho([\![e_1, e_2]\!]) = [\rho(e_1), \rho(e_2)]$,

C3) $[\![e_1, fe_2]\!] = f[\![e_1, e_2]\!] + (\rho(e_1)f)e_2$,

C4) $\rho(e_1)\langle e_2, e_3 \rangle = \langle [\![e_1, e_2]\!], e_3 \rangle + \langle e_2, [\![e_1, e_3]\!] \rangle$,

C5) $[\![e_1, e_1]\!] = \mathcal{D}\langle e_1, e_1 \rangle$.

Here we denote $\mathcal{D} = \frac{1}{2}\rho^* \circ d : C^\infty(M) \to \Gamma(E)$, upon identifying E with E^* using the bilinear form.

Example 4.4.2 (Exact Courant algebroids)**.** Let M be a manifold, and H a closed 3-form on M. The vector bundle $TM \oplus T^*M$ acquires the structure of a Courant algebroid, as follows [ŠW01]. The bilinear pairing is

$$\langle X_1 + \xi_1, X_2 + \xi_2 \rangle = \xi_2(X_1) + \xi_1(X_2),$$

where $X_i + \xi_i \in \Gamma(TM \oplus TM^*)$, the anchor is the first projection, and the bracket is

$$[\![X_1 + \xi_1, X_2 + \xi_2]\!]_H = [X_1, X_2] + \mathcal{L}_{X_1}\xi_2 - i_{X_2}d\xi_1 + i_{X_2}i_{X_1}H. \tag{4.10}$$

We denote this Courant algebroid by $(TM \oplus T^*M)_H$ (up to isomorphism it depends only on the cohomology class of H). When $H = 0$ this is known as standard Courant algebroid structure.

Remark 4.4.3. For any $\xi \in \Gamma(E)$, the map $ad_\xi := [\![\xi, \cdot]\!] \colon \Gamma(E) \to \Gamma(E)$ is an infinitesimal automorphism of the Courant algebroid E. Assuming that the vector field $\rho(\xi)$ is complete, ad_ξ integrates to a 1-parameter group of automorphisms of the Courant algebroid E, which we denote by $e^{t\, ad_\xi}$. For instance, if E is the H-twisted Courant algebroid as in example 4.4.2, and we write $\xi = (X, \eta) \in \Gamma(TM \oplus T^*M)$, the 1-parameter group of automorphisms reads $e^{t\, ad_\xi} = (\varphi_t)_* \circ e^{B_t}$. Here $(\varphi_t)_*$ is the tangent-cotangent lift of the flow φ_t of X, and e^{B_t} is the so-called gauge transformation by the 2-form $B_t := \int_0^t (\varphi_s)^*(d\eta - \iota_X H)\, ds$ [Hu07, §2.2][Gua11, §2.2].

Recall that a *Lie algebroid* is a vector bundle $A \to M$ together with a Lie bracket $[\cdot, \cdot]$ on the sections $\Gamma(A)$ and a vector bundle map $\rho \colon A \to TM$ (called anchor), which are compatible in the sense that $[a_1, fa_2] = \rho(a_1)f \cdot a_2 + f[a_1, a_2]$ for all sections a_1, a_2 and all $f \in C^\infty(M)$. The prototypical example is $A = TM$, and indeed Lie algebroids can be regarded as "generalized tangent bundles". Notice that at any point p, the bracket makes $\ker(\rho_p)$ into a Lie algebra, called isotropy Lie algebra. The Lie bracket and anchor of a Lie algebroid can be equivalently encoded by a degree 1 derivation $d_A \colon \Gamma(\wedge^\bullet A^*) \to \Gamma(\wedge^{\bullet+1} A^*)$ satisfying $(d_A)^2 = 0$, called Lie algebroid differential, and defined by a formula analogous to the one for the de Rham differential on differential forms on a manifold.

Example 4.4.4 (Twisted doubles)**.** We generalize example 4.4.2 replacing the tangent bundle TM with any Lie algebroid. Let B be a Lie algebroid over M, and $H \in \Gamma(\wedge^3 B^*)$ such that $d_B H = 0$. One then obtains a Courant algebroid structure on $B \oplus B^*$, with anchor given by the one of B (thus vanishing on B^*), and with bracket given as[2] in (4.10). Notice that the natural symmetric

[2]For this purpose, replace the Lie derivative appearing in eq. (4.10) with Cartan's formula $\mathcal{L}_a \xi := \iota_a d_B \xi + d_B \iota_a \xi$ for all $a \in \Gamma(B)$ and $\xi \in \Gamma(B^*)$.

pairing on the fibers has split signature. We denote this Courant algebroid by $(B \oplus B^*)_H$. When $H = 0$, this Courant algebroid is known as the double of the Lie bialgebroid (B, B^*) (where the latter is endowed with the trivial Lie algebroid structure).

4.4.2 Dirac structures

Definition 4.4.5. Let $E \to M$ be a Courant algebroid. A *Dirac structure* [Cou90] is a subbundle $A \subseteq E$ which is Lagrangian w.r.t. the pairing (i.e. $A^\perp = A$), and which is involutive w.r.t. the Courant bracket.

Remark 4.4.6. Notice that if $A \subseteq E$ is a Dirac structure, then the restrictions to A of the anchor and of the Courant bracket make A into a Lie algebroid.

Example 4.4.7. Following [ŠW01], we present two classes of Dirac structure for the Courant algebroid $(TM \oplus T^*M)_H$ of example 4.4.2, where H is a closed 3-form on M.

a) Let ω be a 2-form on M, and consider the associated vector bundle map $\omega^\sharp\colon TM \to T^*M, X \mapsto \iota_X\omega$. Then graph$(\omega^\sharp)$ is a Lagrangian subbundle of $TM \oplus T^*M$. It is a Dirac structure in $(TM \oplus T^*M)_H$ precisely when $d\omega = -H$.

b) Let π be a bivector field on M, and consider $\pi^\sharp\colon T^*M \to TM, \xi \mapsto \iota_\xi\pi$. Then graph$(\pi^\sharp)$ is a Lagrangian subbundle. It is a Dirac structure in $(TM \oplus T^*M)_H$ precisely when π is a *H-twisted Poisson structure*, meaning that
$$[\pi, \pi] = 2 \wedge^3 \pi^\sharp(H). \tag{4.11}$$

Example 4.4.8. Generalizing example 4.4.7, let B be a Lie algebroid over M, and $H \in \Gamma(\wedge^3 B^*)$ such that $d_B H = 0$. Let $\pi \in \Gamma(\wedge^2 B)$ such that $[\pi, \pi]_B + 2 \wedge^3 \pi^\sharp(H) = 0$. Then graph$(\pi)$ is a Dirac structure in the Courant algebroid $(B \oplus B^*)_H$ defined in remark 4.4.4.

In particular, $0 \oplus B^*$ is a Dirac structure. When the twist H is not exact, then this Dirac structure does not admit any Dirac complement. Indeed, any such complement would be the graph of an element $\Omega \in \Gamma(\wedge^2 B^*)$ satisfying $-d_B\Omega = H$, yielding a contradiction. (In particular, $B \oplus 0$ is not a Dirac structure).

Remark 4.4.9. Notice that for any $\omega \in \Gamma(\wedge^2 B^*)$, there is an isomorphism of Courant algebroids given by
$$\exp(\omega^\sharp) : (B \oplus B^*)_H \to (B \oplus B^*)_{H-d\omega}, \quad \exp(\omega^\sharp)(X + \alpha) = X + \alpha + \iota_X\omega.$$

This was observed in [Gua11, §2.2] for $B = TM$. In particular, when H is exact, we have $(B \oplus B^*)_H \cong (B \oplus B^*)_0$.

Remark 4.4.10. A Courant algebroid over a manifold M induces a partition of M into immersed submanifolds of varying dimension (called leaves) which are tangent to the image of the anchor map. The same applies for Dirac structures.

4.4.3 Deformations of Dirac structures

Let $E \to M$ be a Courant algebroid, and let $A \subseteq E$ be a Dirac structure. In other to give a description of the Dirac structures nearby A, we make an auxiliary choice of Lagrangian complement K (so $E = A \oplus K$ as vector bundles), and express the Courant algebroid structure in terms of A and K.

Remark 4.4.11. A Lagrangian complement of A exists if and only if E is of even rank $2n$ and the pairing has signature (n, n) (see e.g. [KW07, Corollary 4.4]).

Since the bracket $[\cdot,\cdot]_A := [\![\cdot,\cdot]\!]|_A$ and the bundle map $\rho|_A \colon A \to TM$ make A into a Lie algebroid (see remark 4.4.6), we denote by d_A the corresponding Lie algebroid differential (it squares to zero).

Identify $K \cong A^*$ via the pairing on the fibers of E, i.e. via $K \xrightarrow{\simeq} A^*$, $u \longmapsto \langle u, \cdot \rangle|_A$. Notice that A^* is usually not a Dirac structure. Similarly to the above, the restriction $[\eta_1, \eta_2]_{A^*} := pr_{A^*}([\![(0,\eta_1),(0,\eta_2)]\!])$ on $\Gamma(A^*)$ and the bundle map $\rho|_{A^*} : A^* \to TM$ allow one to write down a degree 1 derivation d_{A^*} of $\Gamma(\wedge^\bullet A)$, which generally does not square to zero.

Consider also the map

$$\Gamma(\wedge^2 A^*) \to \Gamma(A)\ ,\ \eta_1 \wedge \eta_2 \mapsto pr_A([\![(0,\eta_1),(0,\eta_2)]\!]),$$

which measures the failure of A^* to be a Dirac structure, and view it as an element $\Psi \in \Gamma(\wedge^3 A)$.

From Ψ, $(A, [\cdot,\cdot]_A, \rho|_A)$, and $(A^*, [\cdot,\cdot]_{A^*}, \rho|_{A^*})$ one can reconstruct the Courant algebroid structure on $E = A \oplus A^*$: the bracket is recovered as

$$[\![(a_1,\eta_1),(a_2,\eta_2)]\!] = \tag{4.12}$$

$$\Big([a_1,a_2]_A + \mathcal{L}_{\eta_1} a_2 - \iota_{\eta_2} d_{A^*} a_1 + \Psi(\eta_1,\eta_2,\cdot)\ ,\ [\eta_1,\eta_2]_{A^*} + \mathcal{L}_{a_1}\eta_2 - \iota_{a_2} d_A \eta_1\Big)$$

and the anchor as $\rho_A + \rho_{A^*} : A \oplus A^* \to TM$ ([Roy, §3.8], see also [KS05, §3.2]).

The statement of [FZ15, Lemma 2.6] reads as follows[3]:

Proposition 4.4.12. *The graded vector space* $\Gamma(\wedge^\bullet A^*)[2]$ *has an* $L_\infty[1]$*-algebra structure*[4] $\{\mu_k\}$, *whose only non-trivial multibrackets* μ_1, μ_2, μ_3 *are defined as follows:*

$$\mu_1(\alpha[2]) = (d_A\alpha)[2]$$

$$\mu_2(\alpha[2],\beta[2]) = (-1)^{|\alpha|}[\alpha,\beta]_{A^*}[2]$$

$$\mu_3(\alpha[2],\beta[2],\gamma[2]) = -(-1)^{|\beta|}(\alpha^\sharp \wedge \beta^\sharp \wedge \gamma^\sharp)\Psi[2].$$

Further, MC elements $\varepsilon \in \Gamma(\wedge^2 A^*)$ *of this* $L_\infty[1]$*-algebra parametrize Dirac structures* $L \subseteq E$ *that are transverse to* K*, via*

$$L = \mathrm{graph}(\varepsilon^\sharp) = \{a + \iota_a\varepsilon^\sharp \mid \xi \in A\} \subseteq A \oplus A^* \cong E.$$

Here we define $\alpha^\sharp a := \iota_a\alpha$, and

$$(\alpha^\sharp \wedge \beta^\sharp \wedge \gamma^\sharp)(x_1 \wedge x_2 \wedge x_3) = \sum_{\sigma\in S_3}(-1)^\sigma \alpha^\sharp(x_{\sigma(1)}) \wedge \beta^\sharp(x_{\sigma(2)}) \wedge \gamma^\sharp(x_{\sigma(3)}),$$

for all homogeneous $\alpha,\beta,\gamma \in \Gamma(\wedge^\bullet A^*)$ and all $x_i \in \Gamma(A)$.

4.5 Gauge equivalences for Dirac structures

As in §4.4.3, let E be a Courant algebroid, A a Dirac structure, and choose a Lagrangian complement, which we identify with A^* using the pairing (hence $E = A \oplus A^*$ as vector bundles). The main result of this section is Proposition 4.5.2, which gives a geometric description of the gauge equivalence relation that the $L_\infty[1]$-algebra of Proposition 4.4.12 induces on the Dirac structures nearby A.

Recall from rmk 4.4.3 that any element $\xi \in \Gamma(A^*)$ induces a one-parameter group of Courant algebroid automorphisms defined for small t, via $e^{t\, ad_\xi}$, where $ad_\xi = [\![\xi,\cdot]\!]$. We will use repeatedly the following fact, which follows immediately from eq. (4.12):

$$ad_\xi a = \mathcal{L}_\xi a, -\iota_a d_A\xi$$

[3]The global minus in front of the ternary bracket, was erroneously omitted in [FZ15, Lemma 2.6].

[4]This $L_\infty[1]$-algebra structure depends on the choice of K, but it is independent of this choice up to $L_\infty[1]$-isomorphism [GMS20][Tor22].

for all $a \in \Gamma(A)$.

Let $\varepsilon \in \Gamma(\wedge^2 A^*)$ be a Maurer-Cartan element of the $L_\infty[1]$-algebra $\Gamma(\wedge^\bullet A^*)[2]$ of Proposition 4.4.12 (hence graph$(\varepsilon^\sharp)$ is a Dirac structure). For any compactly supported $\xi \in \Gamma(A^*)$, we obtain a smooth one-parameter family of Maurer-Cartan elements, given by the unique solution ε_t of the equation

$$\dot{\varepsilon}_t = -d_A\xi + [\xi, \varepsilon_t]_{A^*} + \frac{1}{2}(\xi^\sharp \wedge \varepsilon_t^\sharp \wedge \varepsilon_t^\sharp)\Psi, \tag{4.13}$$

subject to the initial condition $\varepsilon_0 = \varepsilon$. This is the gauge equation associated to the element $-\xi$ in the $L_\infty[1]$-algebra $\Gamma(\wedge^\bullet A^*)[2]$, cf. equation (4.4).

Remark 4.5.1. We call *gauge equivalence relation* the equivalence relation on Maurer-Cartan elements generated by the following: two Maurer-Cartan elements are related if they can be written as ε_0 and ε_1 as above for some $\xi \in \Gamma(A^*)$. For a comparison of the gauge equivalence relation with other notions in terms of polynomial paths found in the literature, see [DP16, Proposition 9] (see also [KS22, remark 5.22]).

The following proposition states that the 1-parameter family of Dirac structures graph$(\varepsilon_t^\sharp)$ is obtained applying Courant algebroid automorphisms to graph$(\varepsilon^\sharp)$.

Proposition 4.5.2. *Let $E = A \oplus A^*$ be a Courant algebroid as in §4.4.3. Let $\xi \in \Gamma(A^*)$ be compactly supported, and $\varepsilon \in \Gamma(\wedge^2 A^*)$ be a Maurer-Cartan element of the $L_\infty[1]$-algebra of Proposition 4.4.12. Let $\varepsilon_t \in \Gamma(\wedge^2 A^*)$ be determined by the property*

$$\mathrm{graph}(\varepsilon_t^\sharp) = e^{t\, ad_\xi}\, \mathrm{graph}(\varepsilon^\sharp), \tag{4.14}$$

for $t \in \mathbb{R}$ close enough to zero.

Then ε_t is the unique solution of eq. (4.13) *satisfying $\varepsilon_0 = \varepsilon$.*

Remark 4.5.3. Since $pr_A\colon$ graph$(\varepsilon^\sharp) \to A$ is an isomorphism, by continuity we have that $pr_A\colon e^{t\, ad_\xi}\, \mathrm{graph}(\varepsilon^\sharp) \to A$ is an isomorphism for t in an open interval around zero, since ξ is compactly supported.

Proof. Given $a \in A$, we use the notation

$$Y_t^a := e^{t\, ad_\xi}(a + \varepsilon^\sharp a).$$

Then the R.H.S. of eq. (4.14) can be written as $\{Y_t^a : a \in A\}$. So eq. (4.14) is equivalent to the condition that

$$\varepsilon_t^\sharp(pr_A(Y_t^a)) = pr_{A^*}(Y_t^a) \tag{4.15}$$

for all $a \in A$ (here we made use of remark 4.5.3).

Now adopt the notation

$$x_t^a := pr_A(Y_t^a).$$

Notice that

$$ad_\xi Y_t^a = ad_\xi \left(x_t^a + pr_{A^*}(Y_t^a)\right) = ad_\xi(x_t^a) + ad_\xi(\varepsilon_t^\sharp x_t^a) \tag{4.16}$$

using eq. (4.15) in the last equality.

For every section $a \in \Gamma(A)$, we take the time derivative of eq. (4.15), and write it out using eq. (4.12) and (4.16):

- taking the time derivative of the LHS we get

$$\dot{\varepsilon}_t^\sharp(pr_A(Y_t^a)) + \varepsilon_t^\sharp(pr_A(ad_\xi Y_t^a)) = \dot{\varepsilon}_t^\sharp(x_t^a) + \varepsilon_t^\sharp\left((\mathcal{L}_\xi x_t^a) + \Psi(\xi, \varepsilon_t^\sharp x_t^a, \cdot)\right).$$

- Taking the time derivative of the RHS of eq. (4.15), we get

$$pr_{A^*}(ad_\xi Y_t^a) = -\iota_{x_t^a} d_A\xi + [\xi, \varepsilon_t^\sharp x_t^a]_{A^*}.$$

Hence the time derivatives of the LHS and RHS of eq. (4.15) are the same iff

$$\dot{\varepsilon}_t^\sharp(x_t^a) = -\iota_{x_t^a} d_A\xi + [\xi, \varepsilon_t^\sharp x_t^a]_{A^*} - \varepsilon_t^\sharp(\mathcal{L}_\xi x_t^a) - \varepsilon_t^\sharp(\Psi(\xi, \varepsilon_t^\sharp x_t^a, \cdot)). \tag{4.17}$$

Using Lemma 4.5.4 and Lemma 4.5.5 below, we see that the RHS of eq. (4.17) can be written as

$$\iota_{x_t^a}\left(-d_A\xi + [\xi, \varepsilon_t]_{A^*} + \frac{1}{2}(\xi^\sharp \wedge \varepsilon_t^\sharp \wedge \varepsilon_t^\sharp)\Psi\right).$$

This, together with remark 4.5.3, shows that ε_t is the unique solution of the differential equation (4.13) with $\varepsilon_0 = \varepsilon$. □

Lemma 4.5.4. *For all $\xi \in \Gamma(A^*), \varepsilon \in \Gamma(\wedge^2 A^*), a \in \Gamma(A)$ the following identity holds:*

$$\iota_a[\xi, \varepsilon]_{A^*} = [\xi, \varepsilon^\sharp a]_{A^*} - \varepsilon^\sharp(\mathcal{L}_\xi a).$$

Proof. Let Θ denote the degree 3 function on $T^*[2]A[1]$ that, together with the degree -2 Poisson bracket of "functions" $\{\cdot,\cdot\}$, encodes the Courant algebroid structure of $A \oplus A^*$ (see [FZ15, §2.2] and references therein).

Claim: $\{\{\Theta, \xi\}, \varepsilon\}$ equals $[\xi, \varepsilon]_{A^*} \in \Gamma(\wedge^2 A^*)$ plus an element of $\Gamma(A^* \otimes A)$.

To prove the claim, we may assume that $\varepsilon = \eta_1 \wedge \eta_2$ for $\eta_i \in \Gamma(A^*)$. Notice that by definition $\{\{\Theta, \xi\}, \eta_1\} = [\![\xi, \eta_1]\!]$ equals $[x, \eta_1]_{A^*}$ plus an element of $\Gamma(A)$. The claim follows from applying the Leibniz rule to $\{\{\Theta, \xi\}, \eta_1 \cdot \eta_2\}$.

From the claim it follows that for all $b \in \Gamma(A)$,

$$\iota_b \iota_a [\xi, \varepsilon]_{A^*} = \{b, \{a, \{\{\Theta, \xi\}, \varepsilon\}\}\}. \tag{4.18}$$

Now the graded Jacobi identity for $\{\cdot, \cdot\}$ implies

$$\{a, \{\{\Theta, \xi\}, \varepsilon\}\} = -\{\{\Theta, \xi\}, \{\varepsilon, a\}\} - \{\{\{\Theta, \xi\}, a\}, \varepsilon\} = [\![\xi, \iota_a \varepsilon]\!] - \varepsilon^\sharp(\mathcal{L}_\xi a),$$

where to compute the last term we used that the restriction of $\{\cdot, \cdot\}$ to $\Gamma(A^* \otimes A)$ is the pairing, that A^* is isotropic and $pr_A [\![\xi, a]\!] = \mathcal{L}_\xi a$. It follows that the R.H.S. of eq. (4.18) equals $\iota_b \left([\xi, \iota_a \varepsilon]_{A^*} - \varepsilon^\sharp(\mathcal{L}_\xi a) \right)$. □

Lemma 4.5.5. *For all $\xi \in \Gamma(A^*), \varepsilon \in \Gamma(\wedge^2 A^*), a \in \Gamma(A)$ and $\Psi \in \Gamma(\wedge^3 A)$ the following identity holds:*

$$-\varepsilon^\sharp \left(\Psi(\xi, \varepsilon^\sharp a, \,\cdot\,) \right) = \frac{1}{2} \iota_a \left((\xi^\sharp \wedge \varepsilon^\sharp \wedge \varepsilon^\sharp) \Psi \right)$$

Proof. The L.H.S. equals $\Psi(\xi, \varepsilon^\sharp a, \varepsilon^\sharp \cdot)$. For the R.H.S., we may assume that Ψ is decomposible, i.e. $\Psi = x_1 \wedge x_2 \wedge x_3$ for $x_i \in \Gamma(A)$. We then compute

$$(\xi^\sharp \wedge \varepsilon^\sharp \wedge \varepsilon^\sharp) \Psi = 2 \xi^\sharp(x_1) \cdot \varepsilon^\sharp x_2 \wedge \varepsilon^\sharp x_3 + \textit{cycl. perm.},$$

and using the relation $\langle \varepsilon^\sharp x_2, a \rangle = -\langle x_2, \varepsilon^\sharp a \rangle$ it follows that

$$\iota_a \left((\xi^\sharp \wedge \varepsilon^\sharp \wedge \varepsilon^\sharp) \Psi \right) = 2 \Psi(\xi, \varepsilon^\sharp a, \varepsilon^\sharp \cdot).$$

□

Remark 4.5.6. The statement of Proposition 4.5.2 admits a version in which $\xi \in \Gamma(A^*)$ is replaced by a smooth 1-parameter family of elements of $\Gamma(A^*)$, providing a unified approach to the geometric characterization of the equivalences of various kinds of geometric structures (e.g. the foliations and pre-symplectic structures worked out in [SZ20]).

4.6 An application: Stability of fixed points of Dirac structures

We start with a definition:

Definition 4.6.1. Consider a Courant algebroid E with anchor map $\rho \colon E \to TM$ and a Dirac structure A. We say that a point $p \in M$ is a *fixed point of A* whenever $\rho(A_p) = 0$.

In this section we obtain a stability criterion for fixed points of Dirac structures. We do so applying suitably Theorem 4.3.1; this yields Proposition 4.6.2, which we then express in more geometric and explicit terms as Theorem 4.6.7.

4.6.1 Applying Theorem 4.3.1

Assume that we are in the setup of §4.4.3: $E \to M$ is a Courant algebroid, $A \subseteq E$ a Dirac structure. Make an auxiliary choice of Lagrangian complement to A, identify the complement with A^* via the pairing, and denote by ρ_{A^*} the restriction of the anchor of E to A^*. Suppose that $p \in M$ is a fixed point of the Dirac structure A, i.e. $\rho(A_p) = 0$. We want to apply Theorem 4.3.1 to the following data:

i) We take the $L_\infty[1]$-algebra $V = \Gamma(\wedge^\bullet A^*)[2]$, with the brackets μ_1, μ_2, μ_3 as in Proposition 4.4.12. We equip V^{-1} with the C^∞-topology, V^0 with the C^1-topology and V^1 with the C^0-topology.

ii) We take the $L_\infty[1]$-subalgebra $W \subseteq V$ defined by

$$W^i := \{\Lambda \in \Gamma(\wedge^{i+2} A^*) \mid \Lambda_p \in \wedge^{i+2} \ker((\rho_{A^*})_p)\}. \tag{4.19}$$

In Lemma 4.8.1 we check that W indeed is an $L_\infty[1]$-subalgebra and that $W^i \subseteq V^i$ is closed for $i = -1, 0, 1$. Notice that the Maurer-Cartan elements of W are precisely those $\varepsilon \in \Gamma(\wedge^2 A^*)$, such that graph$(\varepsilon^\sharp)$ is Dirac, and $p \in M$ is a fixed point of graph$(\varepsilon^\sharp)$.

iii) For $i = -1, 0$ we pick splittings $\sigma_i : V^i/W^i \to V^i = \Gamma(\wedge^{i+2} A^*)$ consisting of compactly supported sections. This is possible since V^i/W^i are finite-dimensional vector spaces.

iv) As Maurer-Cartan element in W^0 we pick $Q = 0$. Notice that we are considering Maurer-Cartan elements near 0, which correspond to Dirac structures near A.

These choices satisfy the properties required just before Theorem 4.3.1:

a) As pointed out in remark 4.3.3, the continuity of the multibrackets was required in order to make the map R in equation (4.9) continuous. However, Lemma 4.3.4 provides an alternative condition for R to be continuous. We therefore instead check that the conditions of Lemma 4.3.4 are satisfied.
Note that the values of μ_1 and μ_2 in a point $q \in M$ only depend on the first jet of the arguments in q. Moreover, the value of μ_3 in a point only depends on the values of the arguments in q. Consequently, $F = I_p^2\Gamma(\wedge^2 A)$ satisfies the assumptions of Lemma 4.3.4.

b) By the choice of lifts to compactly supported sections in iii) above, the gauge action exists for all $t \in \mathbb{R}$: for any $\xi \in \Gamma(A^*)$ the action $e^{t\,ad_\xi}$

is defined as long as the flow of $\rho(\xi) \in \mathfrak{X}(M)$ is defined. As U we can therefore take any open neighborhood of the origin in V^{-1}/W^{-1}. The continuity and smoothness assertions of the gauge action follow from a standard argument using the smoothness of $\varepsilon \in \Gamma(\wedge^2 A^*)$ and the fact that the topology on the V^i is defined by uniform convergence of some jet of ε on compact sets.

c) Note that an element $\varepsilon \in V^0$ is Maurer-Cartan if and only if its graph is involutive, by Proposition 4.4.12. As the gauge action is by Courant algebroid automorphisms (see Proposition 4.5.2), involutivity is preserved.

We apply Theorem 4.3.1 to the data i)-iv) above. In doing so we invoke Proposition 4.5.2, and we use that $\mu_1 = d_A$. We further use the isomorphism of chain complexes $(V/W, \overline{\mu_1}) \cong \left(\frac{\wedge^\bullet A_p^*}{\wedge^\bullet \ker((\rho_{A^*})_p)}[2], \overline{d_A}\right)$ given by evaluation at p (hence the differential $\overline{d_A}$ is computed extending to an element of $\Gamma(\wedge^\bullet A^*)$, applying d_A and evaluating at p). We obtain:

Proposition 4.6.2. *Let $E \to M$ be a Courant algebroid with anchor ρ, and let $A \subseteq E$ be a Dirac structure. Choose a Lagrangian complement, which we canonically identify with A^* via the pairing. Let $p \in M$ be a fixed point of the Dirac structure A, i.e. $\rho(A_p) = 0$. Assume that*

$$H^2\left(\frac{\wedge^\bullet A_p^*}{\wedge^\bullet \ker((\rho_{A^*})_p)}, \overline{d_A}\right) = 0.$$

Then there exists an C^1-open neighborhood $\mathcal{U}$ of $0 \in \Gamma(\wedge^2 A^*)$, such that for any $\varepsilon \in \mathcal{U}$ for which $\operatorname{graph}(\varepsilon^\sharp)$ is Dirac, the following holds: there is a smooth family $I \subseteq A_p^*/\ker((\rho_{A^*})_p)$, parametrized by a neighborhood of

$$0 \in \ker\left(\overline{d_A} : \frac{A_p^*}{\ker((\rho_{A^*})_p)} \to \frac{\wedge^2 A_p^*}{\wedge^2 \ker((\rho_{A^*})_p)}\right),$$

with the property that $x \in I$ implies:

$$p \text{ is a fixed point of } e^{ad_{\sigma_{-1}(x)}}(\operatorname{graph}(\varepsilon^\sharp)). \tag{4.20}$$

Here $\sigma_{-1} \colon \frac{A_p^*}{\ker((\rho_{A^*})_p)} \to \Gamma(A^*)$ is a fixed splitting taking values in compactly supported sections.

Remark 4.6.3. Instead of W defined by equation (4.19), one could take $\widetilde{W}^\bullet := I_p\Gamma(\wedge^{\bullet+2}A^*)$. Notice that Maurer-Cartan elements in $\widetilde{W}$ correspond to

Dirac structures which coincide with A at p. Applying Theorem 4.3.1 to this data, one obtains the following statement: assume the vanishing of

$$H^2(\Gamma(\wedge^\bullet A^*)/I_p\Gamma(\wedge^\bullet A^*)) \cong H^2(\wedge^\bullet A_p^*, \overline{d_A}).$$

Then for any Dirac structure L near A, there is a family $I \subseteq A_p^*$, parametrized by a neighborhood of

$$0 \in \ker(\overline{d_A} : A_p^* \to \wedge^2 A_p^*),$$

with the property that $x \in I$ implies:

$$(e^{ad_{\sigma_{-1}(x)}}(L))_p = A_p.$$

From this it follows that p is a fixed point of $e^{ad_{\sigma_{-1}(x)}}(L)$, but in general the converse does not hold (see remark 4.7.5 later on).

4.6.2 A geometric restatement

In this subsection we rephrase the hypotheses and the conclusions of Proposition 4.6.2, obtaining in Theorem 4.6.7 a geometric statement which does not make reference to any choice of Lagrangian complements.

Expressing the conclusion of Proposition 4.6.2 in terms of graph($\varepsilon^\sharp$), we see that graph($\varepsilon^\sharp$) has a fixed point nearby p:

Lemma 4.6.4. *Let $\varepsilon \in \Gamma(\wedge^2 A^*)$ and $x \in I$ be as in Proposition 4.6.2. Then* (4.20) *holds if and only if:*

$$\phi_{-\rho(\sigma_{-1}(x))}(p) \text{ is a fixed point of } \varepsilon.$$

Here $\phi_{-\rho(\sigma_{-1}(x))}$ denotes the time-1 flow of the vector field $-\rho(\sigma_{-1}(x))$ on M.

Proof. Write $\xi := \sigma_{-1}(x)$. Since e^{-ad_ξ} is a Courant algebroid automorphism, we have $\rho \circ e^{-ad_\xi} = (\phi_{-\rho(\xi)})_* \circ \rho$. where $\phi_{-\rho(\xi)}$ is the time one flow of the vector field $-\rho(\xi)$. Hence for all $Y \in E_p$ we have

$$\rho_{\phi_{-\rho(\xi)}(p)}(e^{-ad_\xi}Y) = 0 \text{ iff } \rho_p(Y) = 0.$$

Applying this to all $Y \in e^{ad_\xi}$ graph($\varepsilon^\sharp$) the conclusion follows. □

We address how the fixed points of Lemma 4.6.4 depend on the parameters.

Lemma 4.6.5. *The map*

$$\Phi\colon A_p^*/\ker(\rho_{A_p^*}) \to N, \;\; y \mapsto \phi_{-\rho(\sigma_{-1}(y))}(p),$$

restricted to a suitable neighborhood of the origin, is a diffeomorphism onto its image. Here N is the leaf of the Courant algebroid through p.

Proof. The anchor at p induces a linear isomorphism

$$A_p^*/\ker(\rho_{A_p^*}) \to \rho(A_p^*) = \rho(E_p) \tag{4.21}$$

onto T_pN. (The equality holds since $\rho(A_p) = \{0\}$). In terms of the splitting σ_{-1}, the above isomorphism is $y \mapsto \rho_{A^*}(\sigma_{-1}(y))|_p$. Composing first with $-Id_{T_pN}$ and then with the map Ψ obtained applying Lemma 4.8.2, we obtain exactly Φ. □

We now rephrase the hypotheses of Proposition 4.6.2, without making reference to the choice of Lagrangian complement.

Lemma 4.6.6. *Consider the Lie algebra $\mathfrak{g} := A_p$, and denote by $d_\mathfrak{g}$ its Chevalley-Eilenberg differential.*

i) The subspace

$$\mathfrak{h} := (\ker(\rho|_{E_p}))^\perp \tag{4.22}$$

is a Lie ideal of $\mathfrak{g}$. Here the orthogonal is taken w.r.t. the symmetric pairing.

ii) The cochain complex appearing in Proposition 4.6.2 agrees with $\left(\frac{\wedge^\bullet \mathfrak{g}^*}{\wedge^\bullet \mathfrak{h}^\circ}, \overline{d_\mathfrak{g}}\right)$. In particular, it is independent of the choice of Lagrangian complement to A.

Proof. We first motivate the definition of $\mathfrak{h}$. For any Lagrangian complement K of A, recall that we make use of the identification $K \cong A^*, k \mapsto \langle k, \cdot\rangle|_A$. Using that A_p is Lagrangian and that the anchor vanishes on A_p, one can see that under this identification, $\ker(\rho|_{K_p})$ is mapped to

$$\{\langle e, \cdot\rangle|_{A_p} : e \in \ker(\rho|_{E_p})\} = \mathfrak{h}^\circ. \tag{4.23}$$

The equality holds because the annihilator of the l.h.s of (4.23) is given by $\ker(\rho|_{E_p}))^\perp \cap A_p$, which agrees with $\mathfrak{h}$.

i) To see that $\mathfrak{h}$ is a Lie ideal, we need to check that for any $h \in \mathfrak{h}$ and $a \in A_p$ we have $[a, h] \in (\ker(\rho|_{E_p}))^\perp$. Take $e \in \ker(\rho|_{E_p})$. Extending a, h (respectively e) to sections of A (respectively E), we have

$$\langle e, [\![a, h]\!]\rangle = -\langle [\![a, e]\!], h\rangle + \rho(a)\langle e, h\rangle$$

by property C4) in Definition 4.4.1. This function vanishes at p, since $\ker(\rho|_{E_p})$ is closed under the Courant bracket and since $\rho(a)$ is a vector field vanishing at p.

ii) The Chevalley-Eilenberg differential $d_{\mathfrak{g}}$ preserves $\wedge^\bullet \mathfrak{h}^\circ$, since $\mathfrak{h}$ is a Lie ideal (it suffices to check this for elements of $\mathfrak{h}^\circ$). Hence $d_{\mathfrak{g}}$ descends to a differential on the quotient $\frac{\wedge^\bullet \mathfrak{g}^*}{\wedge^\bullet \mathfrak{h}^\circ}$. Observe that the quotient map

$$(\wedge^\bullet A_p^*, \overline{d_A}) \to \left(\frac{\wedge^\bullet \mathfrak{g}^*}{\wedge^\bullet \mathfrak{h}^\circ}, \overline{d_{\mathfrak{g}}}\right) \tag{4.24}$$

is a surjective chain map with kernel given by $\wedge^\bullet \mathfrak{h}^\circ$. Since $\mathfrak{h}^\circ = \langle \ker(\rho|_{K_p}), \cdot \rangle|_{A_p}$ (see equation (4.23)), the map (4.24) descends to an isomorphism between $\left(\frac{\wedge^\bullet A_p^*}{\wedge^\bullet \langle \ker(\rho|_{K_p}), \cdot \rangle|_{A_p}}, \overline{d_A}\right)$ – as defined just before Proposition 4.6.2 – and $\left(\frac{\wedge^\bullet \mathfrak{g}^*}{\wedge^\bullet \mathfrak{h}^\circ}, \overline{d_{\mathfrak{g}}}\right)$. □

We finally can rephrase Proposition 4.6.2 in a more geometric way, and without making reference to Lagrangian complements.

Theorem 4.6.7 (Stability of fixed points of Dirac structures)**.** *Let $E \to M$ be a Courant algebroid whose pairing has split signature. Let $A \subseteq E$ be a Dirac structure which has a fixed point at $p \in M$, i.e. $\rho(A_p) = 0$. Denote by $\mathfrak{g}$ the Lie algebra A_p, and consider its Lie ideal $\mathfrak{h} := (\ker(\rho|_{E_p}))^\perp$. Assume that*

$$H^2\left(\frac{\wedge^\bullet \mathfrak{g}^*}{\wedge^\bullet \mathfrak{h}^\circ}, \overline{d_{\mathfrak{g}}}\right) = 0.$$

Fix a neighborhood $\widetilde{N}$ of p inside the corresponding leaf of the Courant algebroid.

Then there exists an C^1-open neighborhood $\mathcal{U}$ of A in the space of Dirac structures, such that for any $L \in \mathcal{U}$ there is a submanifold F^L of $\widetilde{N}$ consisting of fixed points of L. The dimension of F^L equals that of $\ker\left(\overline{d_{\mathfrak{g}}} \colon \frac{\mathfrak{g}^*}{\mathfrak{h}^\circ} \to \frac{\wedge^2 \mathfrak{g}^*}{\wedge^2 \mathfrak{h}^\circ}\right)$.

Proof. Since E has split signature, there exists a Lagrangian complement to A, see remark 4.4.11. This allows us to apply Proposition 4.6.2. We do so making the following choices in iii) and b) at the beginning of §4.6: the splitting $\sigma_{-1} \colon \frac{A_p^*}{\ker((\rho_{A^*})_p)} \to \Gamma(A^*)$ takes values in sections which, once restricted to the leaf, are supported in $\widetilde{N}$; the open neighborhood U of the origin in the domain is such that $\Phi|_U$ is a diffeomorphism onto its image, where Φ is the map of Lemma 4.6.5.

The cohomological obstruction that appears in Proposition 4.6.2 is identical to the one of the present theorem, by Lemma 4.6.6.

The conclusions of Proposition 4.6.2 imply those of the present theorem. To see this, notice that any Dirac structure L close enough to A is the graph of some element of $\Gamma(\wedge^2 A^*)$. Consider the submanifold $I \subseteq U$ in that proposition. Using Lemma 4.6.4 and the map Φ of Lemma 4.6.5, it follows that $F^L := \Phi(I)$ is a submanifold of $\widetilde{N}$ consisting of fixed points of L. □

Remark 4.6.8. We have a short exact sequence of cochain complexes

$$\{0\} \to \wedge^\bullet \mathfrak{h}^\circ \to \wedge^\bullet \mathfrak{g}^* \to \frac{\wedge^\bullet \mathfrak{g}^*}{\wedge^\bullet \mathfrak{h}^\circ} \to \{0\}$$

with differentials induced by $d_\mathfrak{g}$; notice that the first complex agrees with the Chevalley-Eilenberg complex of the quotient Lie algebra $\mathfrak{g}/\mathfrak{h}$. A piece of the corresponding long exact sequence in cohomology reads $H^2(\mathfrak{g}) \to H^2\left(\frac{\wedge^\bullet \mathfrak{g}^*}{\wedge^\bullet \mathfrak{h}^\circ}, \overline{d_\mathfrak{g}}\right) \to H^3(\mathfrak{g}/\mathfrak{h})$. In particular, when the Lie algebra cohomology groups $H^2(\mathfrak{g})$ and $H^3(\mathfrak{g}/\mathfrak{h})$ vanish, the obstruction in Theorem 4.6.7 also vanishes.

Remark 4.6.9 (Comparison with stability of Lie algebroids)**.** Recall that every Dirac structure inherits a Lie algebroid structure d_A. As one may expect, the cochain complex appearing in the obstruction in Theorem 4.6.7 does not only depend on the induced Lie algebroid structure of A: Indeed, $\mathfrak{g}^*/\mathfrak{h}^\circ$ has the same dimension as the leaf of the Courant algebroid through the fixed point p, by (4.21). As any Dirac structure near A induces a Lie algebroid structure on A which is near d_A, there is a relation with the stability of a fixed point of the Lie algebroid A, as in [CF10]. This relation is reflected in the cohomological obstructions (see also the text below Theorem 2 in the introduction and Lemma 1.12 of [CF10]). Recall that the cohomological obstruction from [CF10] to the stability of a fixed point of A as a Lie algebroid is given by $H^1(\mathfrak{g}, T_pM)$. Here the action of $\mathfrak{g}$ on T_pM for $x \in \mathfrak{g}, v \in T_pM$ is given by

$$x \cdot v = [\rho_A(x), v].$$

For $k \geq 1$, the map

$$\wedge^k \mathfrak{g}^* \to \wedge^{k-1}\mathfrak{g}^* \otimes T_pM$$

$$\alpha_1 \wedge \cdots \wedge \alpha_k \mapsto \sum_{i=1}^k (-1)^{k-i} \alpha_1 \wedge \cdots \wedge \widehat{\alpha_i} \wedge \cdots \wedge \alpha_k \otimes \rho_{A^*}(\alpha_i)$$

descends to an injective chain map

$$\frac{\wedge^k \mathfrak{g}^*}{\wedge^k \mathfrak{h}^\circ} \to \wedge^{k-1}\mathfrak{g}^* \otimes T_pM. \tag{4.25}$$

For $k = 2$, the induced map in cohomology relates the cohomological obstructions.

In general, this map is neither injective, nor surjective, so vanishing of either cohomological obstruction does not imply vanishing of the other. This is to be expected, because while Dirac structures near A are contained in the Lie algebroid structures near d_A, the equivalences for Dirac structures only allow to move p along the leaf through p of the Courant algebroid E. However, if $\rho : E \to TM$ is surjective, then stability of a fixed point in the realm of Lie algebroids does imply stability of the fixed point of the Dirac structure. This is reflected at the level of obstructions: if ρ is surjective, then the map in (4.25) is injective in cohomology, hence the vanishing of $H^1(\mathfrak{g}, T_pM)$ implies the vanishing of the obstruction in Theorem 4.6.7.

Remark 4.6.10. It would be interesting to investigate whether the statement of Theorem 4.6.7 remains true removing the split-signature condition.

4.7 Examples

In this section we present several examples for Theorem 4.6.7, about the stability of fixed points of Dirac structures. All our examples are of the kind we describe in this remark.

Remark 4.7.1. Let B be a Lie algebroid over M and a pick a closed $H \in \Gamma(\wedge^3 B^*)$, yielding a Courant algebroid $(B \oplus B^*)_H$ as in example 4.4.4. Let $\pi \in \Gamma(\wedge^2 B)$ such that

$$[\pi, \pi]_B + 2(\wedge^3 \pi^\sharp)(H) = 0. \tag{4.26}$$

Then $A := \mathrm{graph}(\pi)$ is a Dirac structure, see example 4.4.8. Let $p \in M$ be a fixed point of the Dirac structure A, i.e. $A_p \subseteq \ker(\rho_B)_p \oplus B_p^*$, or equivalently $\rho_B \circ \pi^\sharp = 0$.

One can compute the obstruction as in Theorem 4.6.7, using $\mathfrak{h} = \{0\} \oplus \ker(\rho_B)_p^\circ \subseteq A_p$. Often however we prefer to compute the obstruction using the characterization given in Proposition 4.6.2, since it yields the differential directly, without the need to make explicit the Lie algebra structure of A_p. A Lagrangian complement to A is $B \oplus \{0\}$, which by the pairing is identified with A^*. Notice that the differential on $\Gamma(\wedge^\bullet A^*) \cong \Gamma(\wedge^\bullet B)$ is $d_B = [\pi, \cdot\,]_B + (\wedge^2 \pi^\sharp \otimes \mathrm{id})(H)(\,\cdot\,)$ by [ŠW01, §3]. Hence the obstruction appearing in the theorem is

$$H^2\left(\frac{\wedge^\bullet B_p}{\wedge^\bullet \ker((\rho_B)_p)}, \overline{[\pi, \cdot\,]_B + (\wedge^2 \pi^\sharp \otimes \mathrm{id})(H)(\,\cdot\,)}\right), \tag{4.27}$$

where the differential is computed extending to sections of $\wedge^\bullet B$ and then evaluating at p.

Corollary 4.7.2. *Fix a closed 3-form $H \in \Omega^3(M)$. Let $\pi \in \Gamma(\wedge^2 TM)$ be an H-twisted Poisson structure. Let $p \in M$ be a point such that $\pi_p = 0$.*

Recall that $\mathfrak{g} := T_p^*M$ carries a Lie algebra structure, defined by $[d_p f, d_p g] = d_p\{f, g\}$ where f, g are functions. If its second Chevalley-Eilenberg cohomology vanishes, i.e. $H^2(\mathfrak{g}) = 0$, then any H-twisted Poisson structure nearby π vanishes along a submanifold of dimension $\dim(H^1(\mathfrak{g}))$.

For $H = 0$, this recovers [CF10, Theorem 1.1] for zero-dimensional leaves, and [DW06, Theorem 1.2] for first order singularities. Note that the obstruction does not depend on H.

Proof. By example 4.4.7 we know that $A := \mathrm{graph}(\pi) \subseteq E_H := (TM \oplus T^*M)_H$ is a Dirac structure. Further, $\pi_p = 0$ means that $p \in M$ is a fixed point of the Dirac structure A.

As Lagrangian complement to A we choose $B = TM \cong A^*$. The complex appearing in (4.27) is just $\wedge^\bullet T_pM$, since the anchor $\rho|_{TM}$ is injective. The differential reads $\overline{[\pi, \cdot\,]}$ (notice that the second summand in (4.27) vanishes, since $\pi_p = 0$).

This is exactly the complex computing the Chevalley-Eilenberg cohomology of the isotropy Lie algebra of A at the point p, and this Lie algebra is the one described in the statement of this corollary. □

Example 4.7.3 (Cartan-Dirac structure)**.** Let G be a Lie group with a bi-invariant, possibly indefinite metric $(\cdot\,,\,\cdot)$ (for instance, a compact Lie group). The Cartan-Dirac structure on G was introduced in [ŠW01, example 5.2], and is a Dirac structure in the twisted Courant algebroid $(TG \oplus T^*G)_{-H}$. Here H is the Cartan 3-form, i.e. the bi-invariant 3-form on G which at the unit element reads $H(u, v, w) := \frac{1}{2}([u, v], w)$ for $u, v, w \in \mathfrak{g} = T_eG$. Explicitly, it is given by $A := \{(v^L - v^R, \frac{1}{2}(v^L + v^R)^\flat) : v \in \mathfrak{g}\}$, where v^L and v^R denote the left-invariant and right-invariant extension, and $\flat$ denotes contraction with the metric.

With the induced Lie algebroid structure, A is isomorphic (over Id_G) to the transformation Lie algebroid associated to the action of G on itself by conjugation. In particular the leaves of the Cartan-Dirac structure A are the conjugacy classes of G. So the unit $e \in G$ is a fixed point of A, and the isotropy Lie algebra of A at e is just $\mathfrak{g}$. By Corollary 4.7.2 we hence know that if $H^2(\mathfrak{g}) = 0$ then, for any neighborhood $U \subseteq G$ of e, there exists a neighborhood of A in the space of Dirac structures consisting of H-twisted Poisson structures with a fixed point near p.

In the preceding corollary and example, the twisting 3-forms H could be chosen such that their cohomology classes are nonzero. However, as the stability problem is local, the only thing that matters is the cohomology class when restricting to a neighborhood of a point. When the restricted twisting is exact, locally there exist a Dirac complement, by example 4.4.8. Below we give an instance where the twisting is not even locally exact, and there is no locally defined Dirac complement. In such a case, [Sin22, Theorem 5.50] does not apply, and one really needs the more general statement we provided in Proposition 4.6.2.

Example 4.7.4. Let $M = \mathbb{R}^4$ with coordinates (x_1, x_2, x_3, x_4), and let Z be the self-crossing hypersurface given by the equation $x_1x_2x_3 = 0$. Let B_Z be the associated c-tangent bundle [MS21]. This is the Lie algebroid whose sections consist of vector fields on $\mathbb{R}^4$ which are tangent to Z; an adapted frame is provided by $\{e_1, e_2, e_3, e_4\}$, where $\rho_{B_Z}(e_i) = x_i\partial_{x_i}$ for $i = 1, 2, 3$ and $\rho_{B_Z}(e_4) = \partial_{x_4}$. Let $\{e^1, e^2, e^3, e^4\}$ denote the dual frame of B_Z^*. Then $\pi \in \Gamma(\wedge^2 B_Z)$ given by

$$\pi = x_4 e_1 \wedge e_4$$

satisfies

$$[\pi, \pi]_{B_Z} = 2 \wedge^3 \pi^\sharp(H), \tag{4.28}$$

for any[5] choice of d_{B_Z}-closed $H \in \Gamma(\wedge^3 B_Z^*)$. Note that the right hand side vanishes, as $\pi^\sharp : B_Z^* \to B_Z$ has rank at most 2. Equation (4.28) implies that $A := \mathrm{graph}(\pi^\sharp)$ is a Dirac structure in the Courant algebroid $(B_Z \oplus B_Z^*)_H$, and $p = 0 \in \mathbb{R}^4$ is a fixed point since π vanishes there.

To compute the cohomological obstruction, we denote

$$\mathfrak{h} := \ker((\rho_{B_Z})_p) = \mathrm{span}_{\mathbb{R}}\{e_1(p), e_2(p), e_3(p)\}.$$

Then the complex appearing in (4.27), in the relevant degrees, can be identified with

$$\mathbb{R}e_4(p) \xrightarrow{[\pi,\cdot]_{B_Z}} \mathfrak{h} \wedge \mathbb{R}e_4(p) \xrightarrow{[\pi,\cdot]_{B_Z}} \wedge^2\mathfrak{h} \wedge \mathbb{R}e_4(p),$$

by using the decomposition $(B_Z)_p = \mathfrak{h} \oplus \mathbb{R}e_4(p)$. Here, the differential should be interpreted as extending an element $v \in \wedge^i\mathfrak{h} \wedge e_4(p)$ to a local section $\widetilde{v} \in \Gamma(\wedge^{i+1} B_Z)$, computing $[\pi, \widetilde{v}]_{B_Z}(p)$ and projecting to the subspace given by $\wedge^{i+1}\mathfrak{h} \wedge \mathbb{R}e_4(p)$. The cohomology of the above complex at $\mathfrak{h} \wedge \mathbb{R}e_4(p)$ vanishes, as one sees using the fact that the frame $\{e^1, e^2, e^3, e^4\}$ of B_Z^* consists of pairwise

[5]For instance $H = e^1 \wedge e^2 \wedge e^3$.

commuting sections. Thus Theorem 4.6.7 implies that any Dirac structure in $(B_Z \oplus B_Z^*)_H$ close to graph$(\pi^\sharp)$ has a fixed point near 0.

Remark 4.7.5. In view of remark 4.6.3, notice that $A_p = (B_Z^*)_p$. However, the vanishing of π as a section of $\Gamma(\wedge^2 B_Z)$ is not stable, as the graph of the c-bivector field $\pi_t = \pi + te_1 \wedge e_2$ is Dirac, but does not coincide with B_Z^* at any point for $t \neq 0$.

Remark 4.7.6 (On the induced Poisson bivector field)**.** Let B be a Lie algebroid over M with anchor ρ, let h be a closed 3-form on M, and let $\pi \in \Gamma(\wedge^2 B)$ satisfying (4.26) for $H := \rho^* h$. Then $\pi_M := (\wedge^2 \rho)\pi$ is an h-twisted Poisson bivector field on M. If p is a fixed point of graph(π), then p is a fixed point of π_M, since $\pi_M^\sharp = \rho \circ \pi^\sharp \circ \rho^*$. Therefore, assuming for simplicity that the anchor ρ is an isomorphism on an open dense set of M, Theorem 4.6.7 implies the following: if (4.27) vanishes, then any h-twisted Poisson bivector field nearby π_M which can be lifted to B, has a fixed point nearby p. For instance, when B is the c-tangent bundle associated to a self-crossing hypersurface Z, this is a statement about h-twisted Poisson bivector field nearby π_M which are tangent to Z. See [Sin22, §5.1.5] for an example in the case $h = 0$.

4.8 Appendix

In this appendix we prove two lemmas needed in the body of the paper.

Assume the setting and notation introduced at the beginning of §4.6. In item ii) there, we stated that a certain subspace W is a closed $L_\infty[1]$-subalgebra of the $L_\infty[1]$-algebra V introduced there. We now prove this fact.

Lemma 4.8.1. *Let $W^i \subseteq V^i$ be defined by*

$$W^i := \{\Lambda \in \Gamma(\wedge^{i+2} A^*) \mid \Lambda_p \in \wedge^{i+2} \ker((\rho_{A^*})_p)\}.$$

Then

1) *$W^i \subseteq V^i$ is a closed subspace for $i = -1, 0, 1$,*
2) *$W = \bigoplus_{i=-2}^{\infty} W^i$ is a $L_\infty[1]$-subalgebra of $(\Gamma(\wedge^{\bullet+2} A^*), \{\mu_k\}_{1 \le k \le 3})$.*

Proof.

1) Recall that the evaluation map $\mathrm{ev}_p : \Gamma(\wedge^{i+2} A^*) \to \wedge^{i+2} A_p^*$ is continuous when the left hand side is equipped with the C^k-topology for some $k \geq 0$. As $\wedge^{i+2} \ker((\rho_{A^*})_p) \subseteq \wedge^{i+2} A_p^*$ is closed, it follows that $W^i = \mathrm{ev}_p^{-1}(\wedge^{i+2} \ker((\rho_{A^*})_p))$ is closed.

2) Notice that W is invariant under wedge product and the multibrackets μ_1, μ_2, μ_3 are graded derivations in each entry [GTZ22, Remark B.2]. Because of this it is sufficient to show that W^{-2} and W^{-1} – the degree components that generate W – are closed under the multibrackets. Since $W^{-2} = V^{-2}$ and by degree reasons, we are actually reduced to showing that for $f \in W^{-2} = C^\infty(M)$ and $X, Y \in W^{-1} = \mathrm{ev}_p^{-1}(\ker((\rho_{A^*})_p))$:

$$\mu_1(f) \in W^{-1}, \quad \mu_1(X) \in W^0, \quad \mu_2(X,Y) \in W^{-1}.$$

We already showed the first two statements in the proof of Lemma 4.6.6 ii).

We show that $\mu_2(X,Y) \in W^{-1}$. As

$$\mu_2(X,Y) = [\![X,Y]\!] - \Psi(X,Y,\cdot),$$

we note that

$$\begin{aligned}\rho_{A^*}(\mu_2(X,Y)) &= \rho([\![X,Y]\!]) - \rho_A(\Psi(X,Y,\cdot)) \\ &= [\rho(X),\rho(Y)] - \rho_A(\Psi(X,Y,\cdot)).\end{aligned}$$

Evaluating the right hand side in p, the first term vanishes because it is the Lie bracket of vector fields $\rho(X)$ and $\rho(Y)$ which vanish in p, while the second term vanishes because ρ_A vanishes at p. This shows that $\mu_2(X,Y) \in W^{-1}$.

□

The following statement is needed in the proof of Lemma 4.6.5. We include a proof for completeness.

Lemma 4.8.2. *Let N be a manifold, p a point, and consider a linear map $X\colon T_pN \to \mathfrak{X}_c(N)$ to the compactly supported vector fields, mapping each vector $v \in T_pN$ to a vector field X^v extending it (i.e. $X^v(p) = v$). Denote by $\phi^1_{X^v}$ the time-1 flow of X^v. Then the map*

$$\Psi\colon T_pN \to N,\ v \to \phi^1_{X^v}(p),$$

when restricted to a suitable neighborhood of the origin, is a diffeomorphism onto its image.

Proof. We can express Ψ in terms of the vector field Y on $N \times T_pN$ defined by $Y(q,v) := X^v(q)$, as follows: $\Psi(v) = pr_N(\phi^1_Y(p,v))$. This description implies that the map Ψ is smooth. The derivative of Ψ at the origin is Id_{T_pM}, as one computes

$$(d_0\Psi)(v) = \frac{d}{dt}\bigg|_{t=0} \Psi(tv) = X^v(p) = v$$

using $\phi^1_{X^{tv}} = \phi^1_{tX^v} = \phi^t_{X^v}$. Hence the statement follows from the inverse function theorem. □

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Chapter 5

Conclusion and outlook

In Chapter 2, we first computed some examples of universal L_∞-algebroids of linear foliations, motivated by the question whether in this case there exists a universal L_∞-algebroid with only a unary and binary bracket. While we showed that for the Lie algebra of endomorphisms preserving a linear symplectic form on a vector space, there exists a universal Lie n-algebroid with a non-zero ternary bracket, this does not give a conclusive answer to the question, as the universal Lie n-algebroid is not unique. It would be interesting to further investigate this question.

We also gave a constructive way to obtain some invariants extracted from a universal L_∞-algebroid of a linear foliation, without needing the universal L_∞-algebroid, in particular the dimension of the vector bundles involved in a geometric resoution of a foliation which is minimal at a fixed point of the foliation. A natural question we would like to further investigate is then whether the whole projective resolution could be obtained in a constructive way.

In Chapters 3 and 4, we proved algebraic theorems about differential graded Lie algebras and $L_\infty[1]$-algebras, giving sufficient criteria for the inclusion of a subalgebra to induce a local surjection on the space of Maurer-Cartan elements up to equivalence. This question arose as the framework behind studying the stability of fixed points of geometric structures that induce singular foliations.

This reformulation paves the way for a variety of questions. In the examples we considered, the property whose stability we studied was associated to a point and was a finite-dimensional condition. The general problem of stability of compact leaves can however also be formulated in this way, by choosing an appropriate differential graded Lie subalgebra, which does not have finite codimension. Moreover, in the case of Lie algebroids and Poisson structures, our cohomological obstruction is consistent with the cohomological obstruction of [CF10]. Further work could extend in two directions, which are closely related:

- Addressing specific stability problems.
- Further develop the general framework.

For specific stability problems, the stability of higher-dimensional leaves of the structures considered in this thesis are natural generalizations. While the algebraic theorems in this thesis do not apply, they nevertheless give a candidate for the cohomological obstruction. Additionally, one can also consider the stability of leaves with extra structure, of which the stability of *symplectic leaves* as in [CF10] is an example. Furthermore, given a geometric structure that induces a singular foliation, it can often be restricted to the leaf. The question is then when the leaf is stable, such that the corresponding restrictions are isomorphic.

Beyond problems that are related to stability of leaves, there are also algebraic applications of Theorem 3.3.20. It can be shown that in particular, the results of [CSS14] and their analogues for finite-dimensional associative algebras can also be recovered from Theorem 3.3.20. These results have natural analogues for Lie algebroids. Furthermore, when considered from a geometric perspective, a natural question given a flat connection on a vector bundle $E \to M$ which is compatible with some G-structure, is when nearby flat connections are also compatible with the G-structure up to equivalence.

In the direction of associative algebras, it would also be interesting to explore applications in operator algebras.

To further develop the general framework, a natural question is to what extent the hypothesis of finite codimension can be relaxed in Theorems 3.3.20 and 4.3.1, in order to give a unified framework to handle the questions mentioned above.

Another direction to extend the general framework to is to consider more general morphisms of $L_\infty[1]$-algebras. This could include passing from the inclusion of a $L_\infty[1]$-subalgebra to a general (strict) map of $L_\infty[1]$-algebras, or to consider general $L_\infty[1]$-morphisms (see for instance [KS22] for a definition), which also induce a map on Maurer-Cartan elements up to equivalence.

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